

# Asymptotic structure of Poynting dominated jets

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## ABSTRACT

In relativistic, Poynting dominated outflows, acceleration and collimation are intimately connected. An important point is that the Lorentz force is nearly compensated by the electric force therefore the acceleration zone spans a large range of scales. We derived the asymptotic equations describing relativistic, axisymmetric MHD flows far beyond the light cylinder. These equations do not contain either intrinsic small scales (like the light cylinder radius) or terms that nearly cancel each other (like the electric and magnetic forces) therefore they could be easily solved numerically. They also suit well for qualitative analysis of the flow and in many cases, they could even be solved analytically or semi-analytically. We show that there are generally two collimation regimes. In the first regime, the residual of the hoop stress and the electric force is counterbalanced by the pressure of the poloidal magnetic field so that at any distance from the source, the structure of the flow is the same as the structure of an appropriate cylindrical equilibrium configuration. In the second regime, the pressure of the poloidal magnetic field is negligible small so that the flow could be conceived as composed from coaxial magnetic loops. In the two collimation regimes, the flow is accelerated in different ways. We study in detail the structure of jets confined by the external pressure with a power law profile. In particular, we obtained simple scalings for the extent of the acceleration zone, for the terminal Lorentz factor and for the collimation angle.

*Subject headings:* (magnetohydrodynamics:) MHD – relativity – galaxies:jets – gamma rays: bursts

## 1. Introduction

Highly collimated, relativistic jets are observed in active galactic nuclei (AGNs), microquasars and gamma-ray bursts (GRBs). According to the most popular model, these outflows are powered hydromagnetically. By analogy with pulsars, it is assumed that the

magnetosphere of a rapidly rotating accretion disk (Lovelace 1976; Blandford 1976) and the black hole itself (Blandford & Znajek 1977) opens into a relativistic wind that transfers the energy away in the form of the Poynting flux. A long debated question is how and where the electro-magnetic energy is transferred to the plasma. The Poynting flux could be transferred to the kinetic energy of the flow by gradual acceleration however, the acceleration strongly depends on the geometry of the flow (Chiueh et al. 1991; Begelman & Li 1994; Vlahakis 2004) so that acceleration and collimation are intimately connected.

General theorems affirm (Heyvaerts & Norman 1989, 2003; Chiueh et al. 1991; Bogovalov 1995) that at the infinity, the flow should collimate to the rotational axis, a good fraction of the electromagnetic energy being converted into the kinetic energy. However, it has been found that without an external confinement, the characteristic collimation/acceleration scale is exponentially large (Eichler 1993; Begelman & Li 1994; Tomimatsu 1994; Beskin et al. 1998; Bogovalov 1998; Chiueh et al. 1998; Bogovalov & Tsinganos 1999). That is why in pulsar winds, the Poynting flux is converted into the plasma energy predominantly via dissipation processes (see, e.g., review by Kirk et al. (2007)). On the other hand, relativistic jets are observed in the sources where interaction of the outflows with the external medium could not be neglected. In accreting systems, the relativistic outflows from the black hole and the internal part of the accretion disc could be confined by the (generally magnetized) wind from the outer parts of the disk. A widely accepted model of long-duration GRBs assumes that a relativistic jet from the collapsing core pushes its way through the stellar envelope. In all these cases the external pressure could be responsible for collimation of Poynting dominated outflows. Moreover, the flow is efficiently accelerated in the collimated outflows so that a significant fraction of the Poynting flux could be eventually converted into the plasma kinetic energy. Note that non-magnetized jets could also be efficiently focused by an ambient medium (Eichler 1982; Peter & Eichler 1995; Levinson & Eichler 2000; Bromberg & Levinson 2007). An advantage of magnetically driven outflows is a relatively low mass load, which naturally leads to highly relativistic velocities.

An explicit solution for the relativistic magnetized wind from the accretion disk was found in the force-free approximation by Blandford (1976). In this solution, the magnetic surfaces are nested paraboloids. Beskin & Nokhrina (2006) generalized this solution to include the inertia forces and showed that the magnetic surfaces are only slightly modified and that the flow is accelerated until the equipartition level. A few self-similar solutions to the relativistic magnetohydrodynamic (MHD) equations were found (Li et al. 1992; Contopoulos 1995; Vlahakis & Königl 2003a,b; Narayan et al. 2007), which resemble outflows from a disk. These solutions also demonstrated that collimation and acceleration could occur at a reasonable, even though large, scale. Numerical simulations support these findings (Komissarov et al. 2007, 2008; Tchekhovskoy et al. 2008).

A crucial assumption in these models is a non-zero magnetic flux threading the disk and the black hole. The total flux should in fact be infinite (going to infinity with the outer disk radius) because it is the pressure of the poloidal field, not the hoop stress, that collimates the outflow (Spruit et al. 1997). Such a field could not be generated in the disk; it should be dragged inward by the accreting material (Bisnovatyi-Kogan & Ruzmaikin 1976; Bisnovatyi-Kogan & Lovelace 2007; Rothstein & Lovelace 2008). Magnetized outflows with the zero net magnetic flux, the so called magnetic towers, were proposed by Lynden-Bell (1996) and then studied both analytically (Lovelace & Romanova 2003; Uzdensky & MacFadyen 2006; Lynden-Bell 2006; Sherwin & Lynden-Bell 2007; Gourgouliatos & Lynden-Bell 2008) and numerically (Lovelace et al. 2002; Kato et al. 2004; Nakamura et al. 2006, 2007). Since there is no large scale magnetic field in this model, the jet is collimated by the pressure of the ambient medium so that an extended outflow surrounding the jet is anyway needed.

It is well known that in relativistic MHD outflows, the acceleration zone spans a large range of scales. This is because the electric force, which is negligibly small in the non-relativistic case, becomes comparable with the Lorentz force and when the flow velocity approaches the speed of light, these two forces nearly cancel each other so that both acceleration and collimation proceed very slowly. Within the light cylinder<sup>1</sup>, the magnetosphere corotates with the central source so that the plasma, which slides along the rotating field lines, could acquire only moderate relativistic velocities. Beyond the light cylinder, the flow is accelerated at least until the velocity exceeds the fast magnetosonic velocity. The fast magnetosonic point is already very far from the light cylinder but in this point, the plasma energy is still well below the Poynting flux (e.g., Camenzind (1986)). The complete transformation of the electro-magnetic to the kinetic energy could occur only at the scale much larger than even the distance to the fast magnetosonic surface. This means that a few different spatial scales are present in the problem, which poses a strong challenge to numerical simulations. On the other hand, multi-scale systems are suitable for asymptotic analysis. In the spirit of the method of matched asymptotic expansions, one can solve the equations in two overlapping domains, namely, in the near zone,  $\Omega r \sim 1$ , where the force-free approximation could be used, and in the far zone, where one can considerably simplify the equations in the limit  $(\Omega r)^{-1} \ll 1$ . Both solutions should be matched in the intermediate region where the flow is still force-free but the condition  $\Omega r \gg 1$  is already fulfilled.

In this paper, we study properties of relativistic jets at the distances much larger than the light cylinder radius. First we obtain the asymptotic equations describing the flow in the limit  $\Omega r \gg 1$ . Far enough from the source, these equations are valid till the axis of the

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<sup>1</sup>In differentially rotating magnetospheres, the surface  $\Omega r = 1$  is not a cylinder but we retain the standard term, which has come from the pulsar theory.

flow so that these equations in fact describe the whole flow in the far zone. We apply the obtained equations to jets confined by an ambient medium. We show that there are two different regimes of the flow collimation and acceleration. In the first regime, the structure of the flow at any distance from the source is the same as in an appropriate cylindrical jet, i.e., the residual between the magnetic hoop stress and the electric force is compensated by the pressure of the poloidal field. We will refer to this regime as to equilibrium collimation in the sense that the flow remains in the cylindrical equilibrium. In the second regime, one can neglect the pressure of the poloidal field so that the dynamics of the flow is the same as in the case of purely toroidal field; this regime will be called non-equilibrium. In different collimation regimes, the acceleration regimes are also different.

We show that while the flow is Poynting dominated, the structure of the jet is governed by a simple ordinary differential equation, which could be easily solved for any distribution of the external pressure. The general theory will be applied to jets with a constant angular velocity propagating in a medium with the pressure decreasing as a power law. We also study the structure of the moderately magnetized core of the jet; such a core is presented near the axis of even Poynting dominated flows because the Poynting flux vanishes at the axis. As the jet propagates, the flow is accelerated and the inner parts of the jet reach equipartition between the kinetic and electromagnetic energy so that the moderately magnetized core expands within the jet. Depending on the profile of the confining pressure, the core could either occupy only internal part of the jet so that the main body of the flow remains Poynting dominated or expand till the periphery of the flow such that the whole jet ceases to be Poynting dominated.

The paper is organized as follows. In the next section, we shortly outline derivation of the basic equations describing relativistic, axisymmetric MHD flows. In Sect. 3, we shortly discuss the boundary conditions and integrals of motions. In Sect. 4, we find asymptotic equations for the flow in the far zone. In Sect. 5, we use the derived equations to develop a technique for finding the structure of collimated, Poynting dominated jets. In Sect. 6, we apply this technique to jets with a constant angular velocity propagating in a medium with the pressure decreasing as a power law. The terminal Lorentz factor of the flow as well the terminal collimation angle, are estimated in Sect. 7. In Sect. 8, we study the structure of the moderately magnetized core of the jet. The obtained results are summarized in Sect. 9.

## 2. Basic equations

For the sake of consistency and in order to introduce notations, let us shortly review the basic theory of relativistic, magnetized winds (Okamoto 1978; Lovelace et al. 1986; Li et al.

1992). Let the plasma be cold, which is a good approximation in the far zone where the flow is already expanded. Then the steady state equation of motion is written as

$$\rho\gamma(\mathbf{v} \cdot \nabla)\gamma\mathbf{v} = \frac{1}{4\pi} [(\nabla \cdot \mathbf{E})\mathbf{E} + (\nabla \times \mathbf{B}) \times \mathbf{B}]; \quad (1)$$

where  $\rho$  is the plasma proper density,  $\gamma$  the Lorentz factor,  $\mathbf{v}$  the plasma velocity; the speed of light is taken to be unity. Here the second pair of Maxwell's equation is already used. The equation of motion should be supplemented by the first pair of Maxwell's equations,

$$\nabla \cdot \mathbf{B} = 0; \quad \nabla \times \mathbf{E} = 0; \quad (2)$$

by the continuity equation,

$$\nabla \cdot (\rho\gamma\mathbf{v}) = 0; \quad (3)$$

and by the condition of flux freezing,

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0. \quad (4)$$

In axisymmetric configurations, the magnetic field is conveniently decomposed into the poloidal and toroidal components,  $\mathbf{B} = \mathbf{B}_p + B_\phi \hat{\phi}$ , the poloidal field being expressed via the flux function

$$\mathbf{B}_p = \frac{1}{r} \nabla \Psi \times \hat{\phi}. \quad (5)$$

Here cylindrical  $(r, \phi, z)$  coordinates are used; hat denotes unite vectors. The condition of flux freezing implies that the flux surfaces are equipotentials, which yields

$$\mathbf{E} = -\Omega(\Psi) \nabla \Psi; \quad (6)$$

where  $\Omega(\Psi)$  is the angular velocity of the field line. This gives a useful relation

$$E = r\Omega B_p. \quad (7)$$

The plasma streams along the flux surfaces so that the flow velocity may also be decomposed into the poloidal and toroidal components,  $\mathbf{v} = v_p \hat{\mathbf{l}} + v_\phi \hat{\phi}$ , where  $\hat{\mathbf{l}}$  is the unit vector along the magnetic surface,

$$\hat{\mathbf{l}} = \hat{\mathbf{n}} \times \hat{\phi}; \quad \hat{\mathbf{n}} = \nabla \Psi / |\nabla \Psi|. \quad (8)$$

The condition of flux freezing yields a relation between the components of the velocity and magnetic field:

$$B_p v_\phi - B_\phi v_p = r\Omega(\Psi) B_p; \quad (9)$$

which implies that the plasma slides along the rotating magnetic field lines. The continuity equation (3) could be integrated, with the aid of Eq. (2), into the conservation law

$$4\pi\rho v_p \gamma = \eta(\Psi) B_p; \quad (10)$$

where the function  $\eta$  describes the distribution of the mass flux at the inlet of the flow.

The three remaining equations are obtained by projecting the equation of motion onto directions  $\hat{\mathbf{l}}$ ,  $\hat{\phi}$  and  $\hat{\mathbf{n}}$ . The first two may be manipulated into the integrals of motion

$$\gamma - \frac{r\Omega B_\phi}{\eta} = \mu(\Psi); \quad (11)$$

$$\gamma r v_\phi - \frac{r B_\phi}{\eta} = l(\Psi); \quad (12)$$

representing conservation of the energy and of the angular momentum, correspondingly. Note that the widely used parameter  $\sigma$ , defined as the ratio of the Poynting to the matter energy flux, is presented via the basic quantities as

$$\sigma = \frac{\mu - \gamma}{\gamma}. \quad (13)$$

The projection of the equation of motion onto the normal to the flux surface,  $\hat{\mathbf{n}}$ , yields the transfield force-balance equation (the generalized Grad-Shafranov equation)

$$\frac{1}{\mathcal{R}} \left[ \rho \gamma^2 v_p^2 + \frac{E^2 - B_p^2}{4\pi} \right] - \hat{\mathbf{n}} \cdot \nabla \frac{B_p^2}{8\pi} + \frac{1}{r^2} \rho \gamma^2 v_\phi^2 \hat{\mathbf{n}} \cdot \mathbf{r} = \frac{1}{8\pi r^2} \hat{\mathbf{n}} \cdot \nabla [r^2 (B_\phi^2 - E^2)]; \quad (14)$$

where  $\mathcal{R}$  is the local curvature radius of the poloidal field line (defined such that  $\mathcal{R}$  is positive when the flux surface is concave so that the collimation angle decreases),

$$\frac{1}{\mathcal{R}} = -\hat{\mathbf{n}} \cdot (\hat{\mathbf{l}} \cdot \nabla) \hat{\mathbf{l}} = \hat{\mathbf{n}} \cdot [\hat{\mathbf{l}} \times ((\nabla \times \hat{\mathbf{l}}))] = -\hat{\phi} \cdot (\nabla \times \hat{\mathbf{l}}). \quad (15)$$

Eqs. (7), (9), (10), (11), (12) and (14) form a complete set of equation describing cold, axisymmetric MHD flows. This set could be reduced to a pair of equations for  $\Psi$  and  $\gamma$ .

Eliminating  $B_\phi$  from Eqs. (11) and (12), one can express the azimuthal velocity via  $\Psi$  and  $\gamma$  as

$$v_\phi = \frac{1}{\Omega r} \left( 1 - \frac{\mu - \Omega l}{\gamma} \right). \quad (16)$$

Assuming for simplicity that at the origin of the outflow, the rotation velocity is well below the speed of light,  $\Omega r_{\text{in}}, v_{\phi, \text{in}} \ll 1$ , one reduces Eq. (16) to the form

$$v_\phi = \frac{1}{\Omega r} \left( 1 - \frac{\gamma_{\text{in}}}{\gamma} \right); \quad (17)$$

where the index "in" is referred to the parameters of the injected plasma. Substituting this relation into Eq. (9) and eliminating  $B_\phi$  with the aid of Eq. (11), one gets the expression for the poloidal velocity

$$v_p = \frac{r^2 \Omega^2 B_p}{\eta(\mu - \gamma)} \left[ 1 - \frac{1}{\Omega^2 r^2} \left( 1 - \frac{\gamma_{\text{in}}}{\gamma} \right) \right]. \quad (18)$$

Now one can write the identity  $v_p^2 + v_\phi^2 + \gamma^{-2} = 1$  as the Bernoulli equation

$$\frac{\Omega^4 r^4 B_p^2}{\eta^2 (\mu - \gamma)^2} \left[ 1 - \frac{1}{\Omega^2 r^2} \left( 1 - \frac{\gamma_{\text{in}}}{\gamma} \right) \right]^2 + \frac{1}{\Omega^2 r^2} \left( 1 - \frac{\gamma_{\text{in}}}{\gamma} \right)^2 + \frac{1}{\gamma^2} = 1; \quad (19)$$

which connects the Lorentz factor of the flow with the geometry of the flux tube defined by the function  $\Psi$ .

The transfield equation (14) is converted into an equation for  $\Psi$  and  $\gamma$  upon substituting  $E$  from Eq. (6),  $B_\phi$  from Eq. (11),  $\rho$  from Eq. (10),  $v_p$  from Eq. (18) and  $v_\phi$  from Eq. (17). Therefore Eqs. (14) and (19) form a complete set of equations.

### 3. Boundary conditions and integrals of motion

At the inlet of the flow, one should specify the distribution of the poloidal flux or, which is the same, of the poloidal magnetic field  $B_p$ . We are interested in outflows subtending a finite magnetic flux,  $\Psi_0$ , therefore we have to prescribe a boundary condition at the last magnetic surface. If the flow is confined by the pressure of the external medium, the pressure balance condition should be satisfied at the boundary. In the proper plasma frame, the magnetic field is  $B' = (B^2 - E^2)^{1/2} = (B_\phi^2 + (1 - \Omega^2 r^2) B_p^2)^{1/2}$ . The condition that the pressure of this field is compensated by the external pressure is written as

$$[B_\phi^2 + (1 - \Omega^2 r^2) B_p^2]_{\Psi(r,z)=\Psi_0} = 8\pi p_{\text{ext}}(r, z); \quad (20)$$

where  $p_{\text{ext}}$  is the pressure of the external medium.

In the cold flow, one has also to prescribe the functions  $\Omega(\Psi)$ ,  $\eta(\Psi)$  and  $\gamma_{\text{in}}(\Psi)$  at the inlet of the flow so that only two integrals of motion,  $l(\Psi)$  and  $\mu(\Psi)$ , remain unknown. Assuming that the rotation velocity is non-relativistic at the origin of the flow, we have eliminated the dependence on  $l$  (see transition from Eq. (16) to Eq. (17)). In the general case, the integral  $l$  may be expressed via  $\gamma_{\text{in}}(\Psi)$ ,  $\Omega(\Psi)$ ,  $\eta(\Psi)$ ,  $\mu(\Psi)$  and  $B_p$  at the inlet of the flow making use of (9), (11) and (12). So one has to find only the energy integral,  $\mu$ . This integral is determined by the condition of the smooth passage of the flow through the singular surfaces, Alfvén and modified fast magnetosonic (Li et al. 1992; Tsinganos et al. 1996; Bogovalov 1997; Vlahakis et al. 2000; Vlahakis & Königl 2003a). In the Poynting dominated outflows, the Alfvén surface coincides with the light cylinder,  $\Omega r = 1$ , whereas the fast magnetosonic surface goes into the far zone  $\Omega r \gg 1$ . Transition through the Alfvén surface could be studied in the force-free approximation, i.e. neglecting the plasma energy and inertia.

The force-free limit of the transfield equation is obtained by taking  $\rho = 0$  in Eq. (14).

Making use of Eq. (7), one finds

$$(\Omega^2 r^2 - 1) \frac{B_p^2}{\mathcal{R}} + \frac{1}{2} \hat{\mathbf{n}} \cdot \nabla [(\Omega^2 r^2 - 1) B_p^2] = \frac{1}{2r^2} \hat{\mathbf{n}} \cdot \nabla (r B_\phi)^2 - \Omega^2 B_p^2 \hat{\mathbf{n}} \cdot \mathbf{r}. \quad (21)$$

In the force-free limit, the energy equation (11) is reduced to the form

$$r B_\phi = 2I(\Psi); \quad 2I(\Psi) = \eta(\Psi) \mu(\Psi) / \Omega(\Psi); \quad (22)$$

which means that the current flows along the magnetic surfaces. Now the force-free balance equation can be recast in the form of a second order elliptical equation for  $\Psi$  (Okamoto 1974), which is sometimes called the pulsar equation. By inspecting Eq. (21), one can see that in the pulsar equation, the second derivatives are multiplied by  $(\Omega^2 r^2 - 1)$  so that the equation is singular at the light surface. The condition of regularity at this surface enables one to fix the poloidal current  $I(\Psi)$  (e.g., Fendt (1997); Contopoulos et al. (1999); Uzdensky (2004, 2005); Lovelace et al. (2006); Timokhin (2006)). Then the energy integral is found just adding the matter energy flux as  $\mu = \gamma_{\text{in}} + 2\Omega I/\eta$ . The first term here is small in the Poynting dominated outflows however, one cannot neglect it close to the axis where the current  $I$  goes to zero ( $I = \pi \int j r dr = (\pi/2) j(r=0) r^2 \rightarrow 0$  as  $r \rightarrow 0$ ).

Note that decreasing of the energy flux towards the axis is the generic property of the Poynting dominated outflows because the poloidal current,  $I(\Psi)$ , always goes to zero at  $\Psi \rightarrow 0$ . Such a "hollow cone" energy distribution accounts, in particular, for a specific morphology of the inner Crab and other pulsar wind nebulae (e.g., review by Kirk et al. (2007)). In any case, the exact shape of  $\mu(\Psi)$  depends on the geometry of the flow close to the source. In this paper, we study the flow in the far zone therefore we assume that this function is given together with other integrals of motion.

We would like only to note that the function  $\mu(\Psi)$  has a universal form close to the axis. The poloidal field remains finite at the axis so that Eq.(5) yields

$$\Psi = \frac{1}{2} r^2 B_p(r=0, z); \quad \Psi \rightarrow 0. \quad (23)$$

Beyond the light surface, the magnetic field becomes predominantly toroidal whereas the flow becomes predominantly poloidal (see the next section), therefore Eq. (9) yields  $B_\phi \approx E = r\Omega B_p$ . Then the second term in the energy equation is written, close to the axis, as  $r\Omega B_\phi/\eta = \Omega^2 r^2 B_p(r=0, z)/\eta = 2[\Omega(0)]^2 \Psi/\eta(0)$ . So close to the axis, the energy integral has the universal form

$$\mu(\Psi) = \gamma_{\text{in}}(0) \left( 1 + \frac{\Psi}{\tilde{\Psi}} \right); \quad \Psi \rightarrow 0; \quad (24)$$

where

$$\tilde{\Psi} = \frac{\gamma_{\text{in}}(0) \eta(0)}{2[\Omega(0)]^2}. \quad (25)$$

Note that the flow is Poynting dominated only at  $\Psi \gg \tilde{\Psi}$ .

An important point is that in outflows with a constant angular velocity, one can assume for the estimates that the energy integral is described by the linear function (24) not only close to the axis but across the jet. Both an analytical solution for the paraboloidal flux surfaces (Blandford 1976; Beskin & Nokhrina 2006) and numerical simulations (Komissarov et al. 2007, 2008; Tchekhovskoy et al. 2008) show that this is a good approximation for such jets.

One can also obtain a quite general estimate for the energy integral taking into account that beyond the light cylinder, Eq. (9) yields  $B_\phi \approx -\Omega r B_p$ , which simply means that each revolution of the source adds to the wind one more magnetic loop. Then the second term in Eq. (11) may be estimated as  $(\Omega r)^2 B_p / \eta$ . Making use of the estimate  $\Psi \approx (1/2)r^2 B_p$  (the coefficient is exact when the poloidal field is homogeneous), one finds finally

$$\mu(\Psi) \approx \gamma_{\text{in}} + \frac{2\Omega^2(\Psi)\Psi}{\eta(\Psi)}. \quad (26)$$

This expression provides a rough estimate for the energy integral for arbitrary  $\Omega(\Psi)$  and  $\eta(\Psi)$ .

## 4. The basic equations in the limit $\Omega r \gg 1$

### 4.1. Expansion in $1/r$ .

We are interested in outflows initially dominated by the Poynting flux. In such outflows, the Alfvénic surface, where  $B_\phi \approx B_p$ , nearly coincides with the light surface  $\Omega r = 1$ . In the far zone,  $\Omega r \gg 1$ , the toroidal field decreases as  $B_\phi \propto 1/r$ , see Eq. (11). The poloidal field decreases as  $1/r^2$  therefore in the far zone, the field is nearly toroidal. The flow in the far zone becomes nearly radial because according to Eq.(17),  $v_\phi \propto 1/r$ . In spite of this, one generally have to retain the terms with  $B_p$  and  $v_\phi$  in the equations. The physical reason is that the hoop stress is nearly compensated by the electric force so that one cannot generally neglect small pressure of the poloidal field. The formal reason is that the leading order terms in Eqs. (19) and (14) are the same, which makes the system nearly degenerate, so that one have to retain smaller order terms.

In the transfield equation (14), the leading order terms are those in the right-hand side because the terms in the left-hand side are small either as  $B_p/B_\phi$  or as  $r/\mathcal{R}$ . In the Bernoulli equation (19), one gets the leading order terms just neglecting the terms with  $1/r$  and  $1/\gamma$ . This yields

$$\eta(\mu - \gamma) = \Omega^2 r^2 B_p; \quad (27)$$

or, according to Eqs. (7) and (11),

$$B_\phi + E = 0. \quad (28)$$

If one substituted this relation into the right-hand side of the transfield equation, one would kill the leading order terms. The correct procedure (Vlahakis 2004) is to expand the Bernoulli equation (19) to the first non-vanishing order in  $1/r$  and  $1/\gamma$  and only then to eliminate the leading order terms from Eq.(14). Expanding Eq. (19) yields

$$B_\phi^2 - E^2 \equiv \left( \frac{\eta(\mu - \gamma)}{\Omega r} \right)^2 - (\Omega r)^2 B_p^2 = \left( \frac{\Omega^2 r^2 + \gamma_{\text{in}}^2}{\gamma^2} - 1 \right) B_p^2. \quad (29)$$

Substituting this relation into the right-hand side of Eq. (14), one gets

$$\frac{1}{\mathcal{R}} \left( \rho \gamma^2 v_p^2 + \frac{E^2 - B_p^2}{4\pi} \right) + \frac{1}{r^2} \left( \frac{B_p^2}{4\pi} + \rho \gamma^2 v_\phi^2 \right) \hat{\mathbf{n}} \cdot \mathbf{r} = \frac{1}{8\pi r^2} \hat{\mathbf{n}} \cdot \nabla \left[ \frac{\Omega^2 r^4 B_p^2}{\gamma^2} \left( 1 + \frac{\gamma_{\text{in}}^2}{\Omega^2 r^2} \right) \right]. \quad (30)$$

In this equation, there are no terms which nearly cancel each other. Therefore one can now retain only terms of the lowest order in  $1/r$  and  $1/\gamma$ , e.g., neglecting  $B_p$  with respect to  $E$  or substituting  $v_p$  by unity. Moreover, one can now use Eq.(28), which is the zeroth order approximation to the Bernoulli equation, in order to further simplify this equation. For example, the expression in the first brackets in the left-hand side could be transformed, with the aid of Eqs. (7), (10) and (11), as

$$\rho \gamma^2 v_p^2 + \frac{E^2 - B_p^2}{4\pi} = \frac{1}{4\pi} (4\pi \rho \gamma^2 v_p - \Omega r B_p B_\phi) = \frac{\mu \eta B_p}{4\pi}. \quad (31)$$

The expression in the second brackets in the left-hand side of Eq.(30) could also be simplified in the same way after substituting  $v_\phi$  from Eq. (17):

$$\begin{aligned} \frac{B_p^2}{4\pi} + \rho \gamma^2 v_\phi^2 &= \frac{1}{4\pi \Omega^2 r^2} \left[ -\Omega r B_p B_\phi + 4\pi \rho \gamma^2 v_p \left( 1 - \frac{\gamma_{\text{in}}}{\gamma} \right)^2 \right] \\ &= \frac{B_p}{4\pi \Omega^2 r^2} \left[ -\Omega r B_\phi + \eta \gamma \left( 1 - \frac{\gamma_{\text{in}}}{\gamma} \right)^2 \right] = \frac{\eta B_p}{4\pi \Omega^2 r^2} \left( \mu - 2\gamma_{\text{in}} + \frac{\gamma_{\text{in}}^2}{\gamma} \right). \end{aligned} \quad (32)$$

We can also use Eq. (27) in the right-hand side of this equation. Eventually one finds

$$\frac{\mu \eta B_p}{\mathcal{R}} + \frac{\eta B_p}{\Omega^2 r^4} \left( \mu - 2\gamma_{\text{in}} + \frac{\gamma_{\text{in}}^2}{\gamma} \right) \hat{\mathbf{n}} \cdot \mathbf{r} = \frac{1}{2r^2} \hat{\mathbf{n}} \cdot \nabla \left[ \frac{\eta^2 (\mu - \gamma)^2}{\Omega^2 \gamma^2} \left( 1 + \frac{\gamma_{\text{in}}^2}{\Omega^2 r^2} \right) \right]. \quad (33)$$

This is the asymptotic transfield equation valid at  $\Omega r \gg 1$ . It may be significantly simplified in specific cases.

#### 4.2. Asymptotic transfield equation in different regimes.

If the flow is initially Poynting dominated, one can neglect the terms with  $\gamma_{\text{in}}$  far enough from the axis,  $\mu, \Omega r \gg \gamma_{\text{in}}$ . Then one comes to the equation obtained, in a different form, by Vlahakis (2004):

$$\mu \eta B_p \left( \frac{1}{\mathcal{R}} + \frac{\hat{\mathbf{n}} \cdot \mathbf{r}}{\Omega^2 r^4} \right) = \frac{1}{2r^2} \hat{\mathbf{n}} \cdot \nabla \frac{\eta^2 (\mu - \gamma)^2}{\Omega^2 \gamma^2}. \quad (34)$$

The solution to this equation describes the main body of the flow but it could not be continued to the axis because it could not satisfy the condition  $\Psi(r=0) = 0$ . Close to the axis, where the Poynting flux decreases according to Eq. (24), the terms with  $\gamma_{\text{in}}$  should be retained therefore one should generally solve the full asymptotic equation (33).

Taking into account that the terms with  $\gamma_{\text{in}}$  play role only close to the axis, where the angular velocity and the injection Lorentz factor could be considered as constants, one can present Eq. (33), with the aid of Eq. (27), in a more convenient form

$$\eta \mu B_p \left[ \frac{1}{\mathcal{R}} + \frac{\hat{\mathbf{n}} \cdot \mathbf{r}}{\Omega^2 r^4} \left( 1 - \frac{2\gamma_{\text{in}}}{\mu} + \frac{\gamma_{\text{in}}^2}{\gamma^2} \right) \right] = \frac{1}{2r^2} \left( 1 + \frac{\gamma_{\text{in}}^2}{\Omega^2 r^2} \right) \hat{\mathbf{n}} \cdot \nabla \frac{\eta^2 (\mu - \gamma)^2}{\Omega^2 \gamma^2}. \quad (35)$$

The solution to this equation could be continued to the axis in spite of the fact that the equation was formally derived at the assumption  $\Omega r \gg 1$ . The reason is that far enough from the center, the light cylinder  $\Omega r = 1$  is well within the matter dominated zone,  $\Psi \ll \tilde{\Psi}$ , where the flow is practically hydrodynamic.

In the most interesting case of collimated flows,  $z \gg r$ , one can take  $\hat{\mathbf{n}} \cdot \mathbf{r} = r$  and  $\hat{\mathbf{n}} \cdot \nabla = \partial/\partial r$ . When looking for the shape of the magnetic surfaces, one can conveniently use the unknown function  $r(\Psi, z)$  instead of  $\Psi(r, z)$ . Then, e.g.,

$$B_p = \frac{1}{r} |\nabla \Psi| \approx \frac{1}{r} \frac{\partial \Psi}{\partial r} = \left( r \frac{\partial r}{\partial \Psi} \right)^{-1}. \quad (36)$$

In the same approximation, the curvature radius may be presented as (note that  $\mathcal{R}$  is defined to be positive for concave surfaces)

$$\frac{1}{\mathcal{R}} = -\frac{\partial^2 r}{\partial z^2}. \quad (37)$$

Now the transfield equation for the collimated flows in the far zone could be written as

$$\eta \mu \left[ -\frac{\partial^2 r}{\partial z^2} + \frac{1}{\Omega^2 r^3} \left( 1 - \frac{2\gamma_{\text{in}}}{\mu} + \frac{\gamma_{\text{in}}^2}{\gamma^2} \right) \right] = \frac{1}{2r} \left( 1 + \frac{\gamma_{\text{in}}^2}{\Omega^2 r^2} \right) \frac{\partial}{\partial \Psi} \frac{\eta^2 (\mu - \gamma)^2}{\Omega^2 \gamma^2}. \quad (38)$$

We believe that this equation suits well to numerical solution because it does not contain terms that nearly cancel each other. In many cases it could even be solved analytically.

For analytical solution, this equation could be conveniently considered in two overlapping domains, namely, in the main body of the jet, where the flow is significantly accelerated so that one can neglect the terms with  $\gamma_{\text{in}}/\gamma$ , and close to the axis, where the flux surfaces are nearly straight so that one can neglect the curvature term  $\partial^2 r/\partial z^2$ . Solutions in these domains are smoothly matched in the intermediate zone.

Close to the axis, where the flux surfaces are nearly straight,

$$\frac{d^2 r}{dz^2} \ll \frac{1}{\Omega^2 r^3} \quad (39)$$

one can neglect the term with the derivative in  $z$  and write the transfield equation as an ordinary differential equation (see also Beskin & Malyskin (2000); Beskin & Nokhrina (2006, 2008))

$$\mu \left( 1 + \frac{\gamma_{\text{in}}^2}{\gamma^2} \right) - 2\gamma_{\text{in}} = \frac{\Omega^2 r^2 + \gamma_{\text{in}}^2}{\Omega \gamma} (\mu - \gamma) \frac{\partial}{\partial \Psi} \frac{\eta(\mu - \gamma)}{\Omega \gamma}. \quad (40)$$

We will analyze it in sect. 8. In some cases, the condition (39) is fulfilled across the whole jet; then the full jet structure is described by the one-dimensional equation, the  $z$  dependence entering only via the boundary conditions.

Note that neglecting the derivative in  $z$  in the the transfield equation, one comes to the equation describing cylindrical equilibria. We will refer to such a situation as an *equilibrium collimation* in the sense that at any  $z$ , the jet structure is the same as in an appropriate equilibrium cylindrical configuration.

At  $\Psi \gg \tilde{\Psi}$ , the plasma is significantly accelerated in the far zone so that the transfield equation is reduced to:

$$2\mu\eta r \left( -\frac{\partial^2 r}{\partial z^2} + \frac{1}{\Omega^2 r^3} \right) = \frac{\partial}{\partial \Psi} \frac{\eta^2(\mu - \gamma)^2}{\Omega^2 \gamma^2}. \quad (41)$$

This equation describes the structure of the main body of the jet. One cannot give a simple physical interpretation of terms in this equation however, one can gain some physical insight considering regimes when different terms dominate. If the condition (39) is fulfilled across the jet, one can neglect the derivative in  $z$  thus coming to a  $\Psi \gg \tilde{\Psi}$  limit of Eq. (40). In this case, the jet as a whole is collimated in the equilibrium regime. On the other hand, in some configurations (and anyway far enough from the axis) the condition opposite to (39) is fulfilled; then the term with the second derivative becomes dominant so that the equation is reduced to

$$-2\mu\eta r \frac{\partial^2 r}{\partial z^2} = \frac{\partial}{\partial \Psi} \frac{\eta^2(\mu - \gamma)^2}{\Omega^2 \gamma^2}. \quad (42)$$

This equation could be directly obtained assuming that the field is purely toroidal whereas the flow is purely poloidal (Lyubarsky & Eichler 2001). Then the flux freezing condition

(4) yields  $B_\phi^2 - E^2 = (B_\phi/\gamma)^2$ . Substituting this relation into the transfield equation (14) and dropping the terms with  $B_p$  and  $v_\phi$ , one comes, in the far zone, to Eq. (42). In this case, the function  $\Psi$  could be considered as a specially normalized stream function,  $\rho\gamma\mathbf{v} \propto (1/r)\nabla\Psi \times \tilde{\phi}$ , which describes the flow lines. The poloidal field,  $B_p$ , and the angular velocity,  $\Omega$ , become just auxiliary quantities formally defined by Eqs. (5) and (6). Note that Eq. (42) does not change under the transformation  $\Psi \rightarrow a\Psi$ ;  $\Omega \rightarrow a^{-1}\Omega$ ;  $\eta \rightarrow a^{-1}\eta$ ; where  $a$  is an arbitrary number. As it follows from Eqs. (5), (6) and (10), neither poloidal velocity nor the electric field change under this transformation whereas  $B_p$  and  $\Omega$  could acquire any values. The flow with the poloidal field and azimuthal velocity neglected could be seen as composed from coaxial magnetic loops moving away and expanding together with the plasma. In this case, the difference between the hoop stress and the electric force is not counterbalanced by the pressure of the poloidal field. Taking into account that the electric field could not compensate the hoop stress completely (in the frame moving with the loop, the  $r$  component of the electric field is zero), one concludes that there is a residual force towards the axis of the flow. This does not mean that the flow immediately converges to the axis because in highly relativistic flows, the residual of the hoop stress and the electric force is small. In any case, we will refer to this situation as a *non-equilibrium collimation*. We will see that different regimes of collimation correspond to different acceleration regimes.

### 4.3. Asymptotic Bernoulli equation and boundary conditions

The transfield equation should be supplemented by the Bernoulli equation. We have already used this equation in the zeroth order in  $1/r$ , Eq. (27), when simplified the transfield equation. However, one should be careful when using this equation in order to find  $\gamma$  because  $\gamma$  turns out to be a small difference between two large terms if the flow is Poynting dominated,  $\mu \gg \gamma$ . Therefore  $\gamma$  could be found from the Bernoulli equation in the form of Eq. (27) only if  $\sigma$  is not too large. Generally one should retain higher order terms and use, e.g., Eq. (29). Without loss of accuracy, this equation could be written as a cubic equation for  $\gamma$  (e.g. Beskin et al. (1998))

$$\mu - \frac{\Omega^2 r^2 B_p}{\eta} - \gamma = \frac{\Omega^2 r^2 B_p}{2\gamma^2 \eta} \left( 1 - \frac{\gamma^2 - \gamma_{\text{in}}^2}{\Omega^2 r^2} \right). \quad (43)$$

This equation is reduced to the zeroth order Bernoulli equation (27) if one could neglect the expression in the right-hand side. This expression is small as compared with  $\mu$  however, it could be neglected only when it is less than  $\gamma$ , i.e. only if  $\gamma^3 \gg \mu$ ;  $(\Omega r)^3 \gg \mu$ . Note that  $\gamma = \mu^{1/3}$  when the flow velocity is equal to the fast magnetosonic velocity (e.g., Camenzind (1986)) so that one can find  $\gamma$  from the zeroth order Bernoulli equation only beyond the fast

magnetosonic point. Well within this point, the Lorentz factor could be found from another limit of Eq. (43):

$$\left(1 - \frac{\Omega^2 r^2 B_p}{\mu \eta} + \frac{1}{2\Omega^2 r^2}\right) \gamma^2 = \frac{1}{2} \left(1 + \frac{\gamma_{\text{in}}}{\Omega^2 r^2}\right). \quad (44)$$

Generally one has to solve the cubic equation (43) so that there is no simple expression for  $\gamma$  valid in the whole far zone. This in fact means that the acceleration regimes inside and outside the fast magnetosonic point could be different. For example, in the split monopole wind, which represents a non-confined wind from a point source, the Lorentz factor grows linearly with the radius until the fast magnetosonic point and then the acceleration becomes logarithmically slow (Beskin et al. 1998).

Note that the asymptotic transfield equation (38) is valid both outside and inside the fast magnetosonic point because it was derived only under the condition  $\Omega r \gg 1$ . In the next section, we show that when considering collimated, Poynting dominated flows, one can avoid finding  $\gamma$  from the Bernoulli equation. In this case, the acceleration regime does not change at the fast magnetosonic point therefore this point will not appear more in this paper.

The asymptotic form of the boundary condition (20) may be found by making use of Eq.(29) and taking into account that  $\gamma \gg \gamma_{\text{in}}$  in the outer parts of the Poynting dominated jet; this yields

$$\left(\frac{\Omega r B_p}{\gamma}\right)_{\Psi(r,z)=\Psi_0}^2 = 8\pi p_{\text{ext}}(z). \quad (45)$$

Taking into account Eq. (27), one could write this condition also as

$$\left(\frac{\eta(\mu - \gamma)}{\Omega r \gamma}\right)_{\Psi(r,z)=\Psi_0}^2 = 8\pi p_{\text{ext}}(z). \quad (46)$$

## 5. The Poynting dominated flow in the far zone

Let us first consider the structure of the Poynting dominated flow,  $\mu \gg \gamma$ . Since the Poynting flux goes to zero at the axis, (see Eq. (24)), this approximation is violated close enough to the axis,  $\Psi \lesssim \tilde{\Psi}$ . Moreover, we will find that the flow is accelerated in such a way that the closer the field line to the axis, the earlier (at a smaller  $z$ ) the flow kinetic energy approaches the total energy. Therefore a  $\sigma \sim 1$  core is anyway presented within the Poynting dominated jet so that the results of this section could not be applied close enough to the axis. In Section 8, we find the structure of the flow close to the axis, which is smoothly matched, at a larger  $r$ , with the solution for the Poynting dominated flow.

### 5.1. The governing equation

Here we study the structure of the flow at  $\Psi \gg \tilde{\Psi}$ , i.e., when the Poynting flux initially exceeded the plasma kinetic energy. In this case, we can use the asymptotic transfield equation in the form (41).

As it was discussed in sect. 4.3, one cannot find a simple expression for  $\gamma$  from the Bernoulli equation in order to substitute it into the transfield equation and obtain a single equation for  $\Psi$ . On the other hand,  $\gamma$  could be easily found from the transfield equation provided the shape of the magnetic surfaces,  $r(\Psi, z)$ , is known. An important point is that in this case, an extra accuracy is generally not necessary because in the transfield equation,  $\gamma$  is not presented as a difference of large terms. A special care should be taken only if the flow becomes nearly radial because the curvature of the flux surfaces is determined in this case by small deviations of the flow lines from the straight lines; this case will be specially addressed in sect. 7.2. In this and the next sections, we will neglect corrections of the order of  $\gamma/\mu$  to the shape of the flux line; then the Bernoulli equation (27) is reduced to

$$\Omega^2 r^2 B_p = \eta \mu; \quad (47)$$

which could be considered, with account of Eq.(36), as an equation for  $r(\Psi, z)$ :

$$\mu \eta \frac{\partial r}{\partial \Psi} = r \Omega^2. \quad (48)$$

The solution to this equation is presented as

$$r = D(z) \Phi(\Psi); \quad \Phi(\Psi) = \sqrt{2} \exp \left( \int_{\tilde{\Psi}}^{\Psi} \frac{\Omega^2 d\Psi}{\mu \eta} \right); \quad (49)$$

where  $D(z)$  is an arbitrary function. One sees that the structure of collimated, Poynting dominated jets is generally self-similar. Recall that this equation is valid only at  $\Psi \gg \tilde{\Psi}$ ; the solution will be continued to the axis in the Section 8. In any case,  $D$  is roughly the radius of the very inner part of the jet,  $\Psi \sim \tilde{\Psi}$ .

Close enough to the axis, one can use Eq. (24) for  $\mu$ , which implies

$$\Phi = \sqrt{1 + \frac{\Psi}{\tilde{\Psi}}}. \quad (50)$$

This means the poloidal magnetic field becomes homogeneous,  $\Psi \propto r^2$ ;  $\partial B_p / \partial r = 0$ , well inside the jet,  $\tilde{\Psi} \ll \Psi \ll \Psi_0$ . Note that when finding the expression (24) for  $\mu$ , we assumed that the poloidal field is homogeneous near the axis so that this result is nothing more

than a consistency check. The same expression for  $\Phi(\Psi)$  is also obtained if one uses the general estimate (26) for  $\mu$ . This is also because the coefficient 2 in (26) corresponds to the homogeneous poloidal field. Another coefficient would result in a power law function  $\Phi(\Psi)$ . Such a strong dependence on  $\mu$  arises because  $\mu$  enters in the exponent.

In order to find the function  $D(z)$ , let us substitute Eq.(49) into the left-hand side of Eq. (41) and integrate the obtained equation between  $\tilde{\Psi}$  and  $\Psi_0$ :

$$-2D \frac{d^2 D}{dz^2} \int_{\tilde{\Psi}}^{\Psi_0} \Phi^2 \mu \eta d\Psi + \frac{2}{D^2} \int_{\tilde{\Psi}}^{\Psi_0} \frac{\mu \eta d\Psi}{\Omega^2 \Phi^2} = \left( \frac{\eta \mu}{\Omega \gamma} \right)_{\Psi=\Psi_0}^2 - \left( \frac{\eta \mu}{\Omega \gamma} \right)_{\Psi=\tilde{\Psi}}^2. \quad (51)$$

Note that the region  $\Psi \sim \Psi_0$  contributes mostly into the integrals therefore we could choose  $\tilde{\Psi}$  as the lower limit of integration even though the solution (49) is no longer valid there. One can also neglect the last term in the right hand side as compared with the first one because it could be checked a posteriori that the expression in the brackets grows with  $r$ . Making use of the boundary condition (46), one reduces the right-hand side of this equation to  $8\pi r^2 p_{\text{ext}} = 4\pi \Phi^2 D^2 p_{\text{ext}}$ . Then one gets the equation for  $D(z)$  in the closed form

$$\frac{d^2 D}{dz^2} \int_{\tilde{\Psi}}^{\Psi_0} \Phi^2 \mu \eta d\Psi - \frac{1}{D^3} \int_{\tilde{\Psi}}^{\Psi_0} \frac{\mu \eta d\Psi}{\Omega^2 \Phi^2} = -4\pi [\Phi(\Psi_0)]^2 p(z) D. \quad (52)$$

This equation could be written in the dimensionless form as

$$\frac{d^2 Y}{dZ^2} - \frac{1}{Y^3} + \beta \mathcal{P}(Z) Y = 0; \quad (53)$$

where

$$Z = \Omega(\Psi_0) z; \quad Y(z) = \alpha^{-1/4} \Omega(\Psi_0) D(z); \quad (54)$$

$$\alpha = [\Omega(\Psi_0)]^2 \left( \int_{\tilde{\Psi}}^{\Psi_0} \frac{\mu \eta d\Psi}{\Omega^2 \Phi^2} \right) \left( \int_{\tilde{\Psi}}^{\Psi_0} \Phi^2 \mu \eta d\Psi \right)^{-1}; \quad (55)$$

$$\beta = 4\pi p_0 \left[ \frac{\Phi(\Psi_0)}{\Omega(\Psi_0)} \right]^2 \left( \int_{\Psi_p}^{\Psi_0} \Phi^2 \mu \eta d\Psi \right)^{-1}. \quad (56)$$

$$p_0 = p_{\text{ext}}(z = 1/\Omega(\Psi_0)); \quad p_0 \mathcal{P}(Z) = p(z). \quad (57)$$

This equation generalizes the equation for the jet radius obtained by Komissarov et al. (2008) as an order of magnitude estimate. We see that this equation is in fact asymptotically exact. Moreover, finding  $Y(Z)$  from this equation, one finds the full structure of the flow. Therefore we will call Eq.(53) the governing equation for Poynting dominated jets.

Note that there is one to one correspondence between the terms in the governing equation and in the original asymptotic transfield equation (41). Namely the pressure term (the

last one) in Eq. (53) comes from the right-hand side of Eq. (41) whereas the first two terms correspond to the terms in the left-hand side. Following the discussion in sect. 4.2, one sees that the collimation is in the equilibrium regime if the second term dominates the first one. Then one immediately finds  $Y(Z) = [\beta\mathcal{P}(Z)]^{-4}$ . Of course neglecting the derivative in the equation, one loses solutions. The lost solutions just describe oscillations with respect to the equilibrium state. If the jet is not very narrow, the term  $Y^{-3}$  becomes negligibly small; then the governing equation becomes linear. In this case the jet is collimated in the non-equilibrium regime. In Sect. 6. we present a more detailed analysis for the case of a power law profile of the external pressure.

The solution to the governing equation (53) could be presented (Polyanin & Zaitsev 2002) in the form  $Y = wy$ , where the auxiliary function,  $w$ , satisfies the linear equation

$$\frac{d^2w}{dZ^2} + \beta\mathcal{P}(Z)w = 0. \quad (58)$$

Then the equation for  $y$  has the first integral

$$\left(w^2 \frac{dy}{dZ}\right)^2 = C_1 - \frac{1}{y^2}; \quad (59)$$

which could be immediately integrated once again. Now the general solution to Eq. (53) is found as

$$Y = w \left[ \frac{1}{C_1} + C_1 \left( C_2 + \int \frac{dZ}{w^2} \right)^2 \right]^{1/2}. \quad (60)$$

In section 6, we present such a solution for the jet with a constant angular velocity confined by the external pressure decreasing as a power law.

## 5.2. Finding the structure of the flow

According to Eq. (49), the flux surfaces are self-similar in the Poynting dominated domain. Having found  $Y$  from the governing equation, one finds the shape of the magnetic surfaces as

$$\Omega(\Psi_0)r(\Psi, Z) = \alpha^{1/4}\Phi(\Psi)Y(Z). \quad (61)$$

Taking into account that the region  $\Psi \sim \Psi_0$  contributes mostly into the integrals in Eqs. (55) and (56), one can estimate the coefficients in the governing equation as

$$\alpha \sim [\Phi(\Psi_0)]^{-4}; \quad \beta \sim \frac{2\pi p_0}{[\Omega(\Psi_0)]^4 \Psi_0^2}. \quad (62)$$

In the last equality, we used the estimate (26). Substituting the obtained estimate for  $\alpha$  into Eq. (61), one finds

$$Y \sim r(\Psi_0)\Omega(\Psi_0); \quad (63)$$

so that  $Y$  is of the order of the dimensionless outer radius of the jet. Making use of the estimate  $\Psi \sim (1/2)r^2B_p$ , one can write

$$\beta \sim \left( \frac{8\pi p_0}{B^2} \right)_{\Omega(\Psi_0)r(\Psi_0)=1}; \quad (64)$$

so that  $\beta$  is of the order of the ratio of the external pressure to the magnetic pressure at the base of the flow.

In order to find the Lorentz factor of the flow, one substitutes Eq. (61) into the left hand side of Eq. (41) and performs integration between  $\tilde{\Psi}$  and  $\Psi$  to obtain

$$-2\sqrt{\alpha}Y \frac{d^2Y}{dZ^2} \int_{\tilde{\Psi}}^{\Psi} \Phi^2 \mu \eta d\Psi + \frac{2\Omega^2(\Psi_0)}{\sqrt{\alpha}Y^2} \int_{\tilde{\Psi}}^{\Psi} \frac{\mu \eta d\Psi}{\Omega^2 \Phi^2} = \left( \frac{\eta \mu}{\Omega \gamma} \right)_{\Psi=\Psi}^2 - \left( \frac{\eta \mu}{\Omega \gamma} \right)_{\Psi=\tilde{\Psi}}^2. \quad (65)$$

Retaining only the first term in the right-hand side, as it was done in Eq. (52), one gets the relation for  $\gamma(\Psi, Z)$  in the closed form

$$\left( \frac{\eta \mu}{\Omega \gamma} \right)^2 = -2\sqrt{\alpha}Y \frac{d^2Y}{dZ^2} \int_{\tilde{\Psi}}^{\Psi} \Phi^2 \mu \eta d\Psi + \frac{2\Omega^2(\Psi_0)}{\sqrt{\alpha}Y^2} \int_{\tilde{\Psi}}^{\Psi} \frac{\mu \eta d\Psi}{\Omega^2 \Phi^2}. \quad (66)$$

Specifically for the periphery of the flow,  $\Psi = \Psi_0$ , one finds, with the aid of Eqs. (53) and (55),

$$\gamma(\Psi_0, Z) = \frac{W}{\sqrt{\beta \mathcal{P}(Z)} Y(Z)}; \quad W = \frac{\eta(\Psi_0)\mu(\Psi_0)}{\sqrt{2[\Omega(\Psi_0)]^3}} \left( \int_{\tilde{\Psi}}^{\Psi_0} \frac{\mu \eta d\Psi}{\Omega^2 \Phi^2} \int_{\tilde{\Psi}}^{\Psi_0} \Phi^2 \mu \eta d\Psi \right)^{-1/4}. \quad (67)$$

With the aid of Eq. (37), one can write Eq. (66) as

$$\frac{1}{\gamma^2} = A \frac{r}{\mathcal{R}} + B \frac{1}{\Omega^2 r^2}; \quad (68)$$

where

$$A(\Psi) = 2 \left[ \frac{\Omega(\Psi)}{\Phi(\Psi)\eta(\Psi)\mu(\Psi)} \right]^2 \int_{\tilde{\Psi}}^{\Psi} \Phi^2 \mu \eta d\Psi; \quad (69)$$

$$B(\Psi) = 2 \left[ \frac{\Omega^2(\Psi)\Phi(\Psi)}{\eta(\Psi)\mu(\Psi)} \right]^2 \int_{\tilde{\Psi}}^{\Psi} \frac{\mu \eta d\Psi}{\Omega^2 \Phi^2}. \quad (70)$$

This equation generalizes the equation obtained by Tchekhovskoy et al. (2008) and by Komissarov et al. (2008). Making use of the estimate (26) for  $\mu$ , one finds that the coefficients  $A$  and  $B$  are

always of the order of unity. In the case of equilibrium collimation, when the condition (39) is fulfilled, one can neglect the first term in the right-hand side, which yields the old-established (Buckley 1977) acceleration regime  $\gamma \propto \Omega r$ . In the opposite limit of non-equilibrium collimation, one comes to the scaling  $\gamma \propto \sqrt{\mathcal{R}/r}$  found recently by Beskin et al. (2004).

## 6. Poynting dominated jet with a constant angular velocity

In this section, we apply the above general method to jets with the constant angular velocity,  $\Omega(\Psi) = \text{const}$ . In this case, one can conveniently use the dimensionless variables

$$X = \Omega r; \quad Z = \Omega z. \quad (71)$$

We also assume that the injection is homogeneous,  $\eta(\Psi) = \text{const}$ .

Note that in this case, one can get simple relations assuming that the energy integral,  $\mu$ , is described by the linear function (24) not only close to the axis but across the jet. Then the poloidal flux is homogeneous, see Eq. (50). The coefficients  $\alpha$  and  $\beta$  defined by Eqs. (55) and (56), correspondingly, are reduced to:

$$\alpha = 3 \left( \frac{\tilde{\Psi}}{\Psi_0} \right)^2; \quad (72)$$

$$\beta = \frac{6\pi p_0}{\Omega^4 \Psi_0^2} = \frac{6\pi p_0}{B_0^2}; \quad (73)$$

where  $B_0 \equiv \Omega^2 \Psi_0$  is the characteristic magnetic field at the light surface. Now the flux surfaces are described by a simple formula

$$X = 3^{1/4} \left( \frac{\Psi}{\Psi_0} \right)^{1/2} Y(Z). \quad (74)$$

The coefficients in Eq. (68) are reduced to  $A = 1/3$ ;  $B = 1$  so that one gets the equation obtained by Tchekhovskoy et al. (2008). In terms of  $Y$  and  $\Psi$ , the equation for the Lorentz factor (66) could now be written in the simple form

$$\frac{\sqrt{3}}{\gamma^2} = -\frac{\Psi}{\Psi_0} Y \frac{d^2 Y}{dZ^2} + \frac{\Psi_0}{\Psi} \frac{1}{Y^2}. \quad (75)$$

One sees that close enough to the axis, the second term in the right-hand side dominates, which yields the acceleration regime

$$\gamma = X. \quad (76)$$

If  $Y \gg \sqrt{Z}$ , so that if collimation is not very strong, the first term could dominate in the main body of the jet,  $\Psi \sim \Psi_0$ . Then the Lorentz factor is determined by the curvature of the magnetic surface, namely

$$\gamma = \sqrt{3} \left( X \frac{d^2 X}{dZ^2} \right)^{-1/2} = \sqrt{3\mathcal{R}/r}. \quad (77)$$

In any case, at  $\Psi = \Psi_0$  one finds, with the aid of Eq. (53),

$$\gamma(\Psi_0, Z) = \frac{3^{1/4}}{\sqrt{\beta \mathcal{P}(Z) Y(Z)}}. \quad (78)$$

Below we present not only general formulae for the parameters of the flow but also simple estimates with the aid of Eqs. (72), (74) and (75).

Let the external pressure be decreasing as

$$\mathcal{P} = \frac{1}{Z^\kappa}. \quad (79)$$

Then the auxiliary equation (58) is solved via the Bessel functions so that the general solution to the governing equation (53) could be found analytically. Taking into account that the governing equation is valid only at large  $Z$ , one can use only the appropriate asymptotics of the solution. Since the asymptotics depends on the sign of  $\kappa - 2$ , let us consider different cases separately.

### 6.1. The case $\kappa < 2$

In this case, a solution to Eq. (58) is presented as

$$w = \sqrt{Z} J_{\frac{1}{2-\kappa}} \left( \frac{2\sqrt{\beta}}{2-\kappa} Z^{1-\kappa/2} \right). \quad (80)$$

At a large  $Z$ , this function is reduced to

$$w = \sqrt{\frac{2-\kappa}{\pi}} \left( \frac{Z^\kappa}{\beta} \right)^{1/4} \cos S; \quad S = \frac{2\sqrt{\beta}}{2-\kappa} Z^{1-\kappa/2} - \frac{4-\kappa}{2-\kappa} \frac{\pi}{4}. \quad (81)$$

Substituting this auxiliary function into Eq. (60) yields the general solution to the governing equation

$$Y = \sqrt{\frac{2-\kappa}{\pi}} \left( \frac{Z^\kappa}{\beta} \right)^{1/4} \left[ \frac{1}{C_1} \cos^2 S + C_1 \left( C_2 \cos S + \frac{\pi}{2-\kappa} \sin S \right)^2 \right]^{1/2} \quad (82)$$

At  $C_1 = (2 - \kappa)/\pi$ ,  $C_2 = 0$ , this solution is reduced to a power law

$$Y = \left( \frac{Z^\kappa}{\beta} \right)^{1/4}; \quad (83)$$

which could be found directly from Eq. (53) by neglecting the first term (Komissarov et al. 2008). One sees that with this solution, the first term in Eq.(53) is much less, at  $\kappa < 2$ , than the second one so that the collimation occurs in the equilibrium regime. The general solution (82) also expands as  $Z^{\kappa/4}$  but very long wave oscillations are superimposed on this expansion, which means that the flow could oscillate around the equilibrium state. Such oscillations are possible if the jet was injected not in the equilibrium state. The amplitude of these oscillations could be found by matching to the near zone solution at  $Z \sim 1$ . The spatial period of these oscillations increases with the distance as  $Z^{\kappa/2}$ .

Taking into account that the governing equation becomes algebraic in the equilibrium regime, one can generalize Eq. (83) to the general pressure distribution provided the pressure decreases not faster than  $z^{-2}$ :

$$Y = (\beta \mathcal{P})^{-1/4}. \quad (84)$$

One can see that the jet expands while the confining pressure decreases. When the jet eventually enters the region with the constant pressure, the jet becomes cylindrical.

The Lorentz factor of the flow is found from Eq. (66). For the smooth expansion described by Eq. (83) one can neglect the first term in the right-hand side, which yields

$$\gamma = \frac{\eta \mu Z^{\kappa/4}}{\Omega^2} \left( \frac{\alpha}{\beta} \right)^{1/4} \left( 2 \int_{\tilde{\Psi}}^{\Psi} \frac{\eta \mu d\Psi}{\Phi^2} \right)^{-1/2}. \quad (85)$$

One sees that in accord with the general analysis in section 5.2, the Lorentz factor of the flow is proportional to the cylindrical radius,  $\gamma \propto X$ . If the energy integral is a linear function of  $\Psi$ , Eq. (24), which is anyway the case well within the jet, this expression is reduced just to  $\gamma = X$ , see Eq. (75). This estimate remains valid also for non-power law pressure distributions when the jet shape is described by Eq. (84). If the jet oscillates with respect to the equilibrium expansion, as is described by Eq. (82), the Lorentz factor also oscillates with respect to that given by Eq. (85).

## 6.2. The case $\kappa = 2$

In this case, a solution to Eq. (58) is

$$w = \begin{cases} \sqrt{Z} \cos S; & S = \sqrt{\beta - 1/4} \ln Z; \quad \beta > 1/4; \\ Z^{(1+\sqrt{1-4\beta})/2}; & \beta < 1/4. \end{cases} \quad (86)$$

Now the general solution to the governing equation is

$$Y = \frac{1}{\sqrt{C_1}} Z^{1/2} \begin{cases} \left[ \cos^2 S + C_1^2 \left( C_2 \cos S + \frac{1}{\sqrt{\beta-1/4}} \sin S \right)^2 \right]^{1/2} & \beta > 1/4; \\ Z^{(1/2)\sqrt{1-4\beta}} \left[ 1 + C_1^2 \left( C_2 - \frac{1}{\sqrt{1-4\beta}Z\sqrt{1-4\beta}} \right)^2 \right]^{1/2} & \beta < 1/4. \end{cases} \quad (87)$$

The  $\beta > 1/4$  solution is similar to that for the  $\kappa < 2$  case. At specially chosen constants,  $C_1 = (\beta - 1/4)^{1/2}$ ,  $C_2 = 0$ , it is reduced to a pure power law (Komissarov et al. 2008),

$$Y = \frac{Z^{1/2}}{(\beta - 1/4)^{1/4}}; \quad (88)$$

whereas generally long wavelength oscillations are superimposed on the overall expansion. With the solution (88), both terms in the left-hand side of the governing equation are comparable so that this case is an intermediate between the equilibrium and non-equilibrium collimation.

At  $\beta < 1/4$ , the solution is reduced, at large  $Z$ , to a power law (Komissarov et al. 2008)

$$Y = CZ^k; \quad k = (1 + \sqrt{1 - 4\beta})/2. \quad (89)$$

Note that  $1/2 < k < 1$  so that the flow is collimated but slower than in the case  $\beta > 1/4$ . The constant  $C$  in this solution is not defined; it could be found only by matching to the near zone solution. If the flow was not collimated at  $Z \sim 1$ , there should be  $C \sim 1$ . This solution could be obtained directly by neglecting the second term in the governing equation (53). This means that at  $\beta < 1/4$ , the collimation is non-equilibrium.

The Lorentz factor of the flow is found from Eq. (66). At  $\beta > 1/4$ , one substitutes the solution (88), which yields the relation

$$\frac{\eta^2 \mu^2 Z}{2\Omega^2 \gamma^2} = \frac{1}{4} \left( \frac{\alpha}{\beta - 1/4} \right)^{1/2} \int_{\tilde{\Psi}}^{\Psi} \eta \mu \Phi^2 d\Psi + \left( \frac{\beta - 1/4}{\alpha} \right)^{1/2} \int_{\tilde{\Psi}}^{\Psi} \frac{\eta \mu}{\Phi^2} d\Psi; \quad (90)$$

which yields  $\gamma \propto \sqrt{Z} \propto X$ . In this relation, the terms in the right-hand side are comparable at  $\Psi \sim \Psi_0$ . When  $\Psi$  decreases, the first term decreases faster therefore well inside the jet one can retain only the second term. Making use of Eqs. (50) and (55), one finds  $\gamma = X$ . So in this case the Lorentz factor of the flow is equal to the dimensionless cylindrical radius, which is the general property of the equilibrium collimation.

When  $\beta < 1/4$ , one substitutes the solution (89) into Eq. (66) to yield the relation

$$\frac{\eta^2 \mu^2}{2\Omega^2 \gamma^2} = \frac{\beta C^2 \sqrt{\alpha}}{Z^{2(1-k)}} \int_{\tilde{\Psi}}^{\Psi} \eta \mu \Phi^2 d\Psi + \frac{1}{\sqrt{\alpha} C^2 Z^{2k}} \int_{\tilde{\Psi}}^{\Psi} \frac{\eta \mu}{\Phi^2} d\Psi. \quad (91)$$

At  $\Psi \sim \Psi_0$ , the first term in the right-hand side dominates the second one; this could be easily seen from Eq. (75), which is the approximate form of Eq. (66). Then one finds

$$\gamma = \frac{\eta\mu}{\Omega C} \left( 2\beta\sqrt{\alpha} \int_{\tilde{\Psi}}^{\Psi} \eta\mu\Phi^2 d\Psi \right)^{-1/2} Z^{1-k}; \quad (92)$$

which reproduces, in this specific case, the scaling  $\gamma \propto \sqrt{\mathcal{R}/r}$  common to the non-equilibrium collimation. For  $\mu$  given by the lineat function (24), this relation is reduced, with the aid of Eqs. (50) and (72) to

$$\gamma = \frac{3^{1/4}}{C} \sqrt{\frac{\Psi_0}{\beta\Psi}} Z^{1-k}. \quad (93)$$

Note that in this case, the Lorentz factor increases, at a fixed  $Z$ , towards the axis. The Lorentz factor increases until at small enough  $\Psi$ , the second term in the right-hand side of Eq. (91) becomes dominant, which means that close enough to the axis, the jet is in pressure equilibrium. In this region, one has

$$\gamma = \alpha^{1/4} C \sqrt{\frac{\Psi}{\tilde{\Psi}}} Z^k = X; \quad (94)$$

as in any equilibrium flow. So at any fixed  $Z$ , the Lorentz factor increases outwards from the axis while the flow is in the pressure equilibrium and then decreases outwards.

The transition from the non-equilibrium to the equilibrium zone occurs at

$$\frac{\Psi}{\Psi_0} = \frac{1}{\sqrt{\beta} C^2 Z^{2k-1}}; \quad (95)$$

when the two terms in the right-hand side of Eq. (91) become equal. Transforming to the coordinate space with the aid of Eqs. (61) and (89), one sees that the transition occurs at the line

$$Z = \sqrt{\beta} X^{-2}. \quad (96)$$

The Lorentz factor of the flow increases as  $\gamma = X$  while the flow remains within the line (96) whereas after the flow enters the non-equilibrium zone, the acceleration proceeds slower, according to Eq. (93).

### 6.3. The case $\kappa > 2$

In this case, a solution to Eq. (58) is presented as

$$w = \sqrt{Z} J_{\frac{1}{\kappa-2}} \left( \frac{2\sqrt{\beta}}{\kappa-2} Z^{1-\kappa/2} \right). \quad (97)$$

The large  $Z$  asymptotics of this function corresponds to the small argument limit of the Bessel function,

$$J_\nu(x) = \frac{1}{\Gamma(\nu+1)} \left(\frac{x}{2}\right)^\nu; \quad (98)$$

where  $\Gamma$  is the gamma-function. Then the auxiliary function,  $w$ , goes to a constant

$$w = \frac{1}{\Gamma\left(\frac{\kappa-1}{\kappa-2}\right)} \left(\frac{\sqrt{\beta}}{\kappa-2}\right)^{1/(\kappa-2)}; \quad (99)$$

whereas the general solution (60) goes, at large  $Z$ , to a linear function

$$Y = \sqrt{C_1} \Gamma\left(\frac{\kappa-1}{\kappa-2}\right) \left(\frac{\kappa-2}{\sqrt{\beta}}\right)^{1/(\kappa-2)} Z; \quad (100)$$

so that the flow becomes radial at large distances.

One sees from Eq. (100) that the flow could be collimated,  $Y \ll Z$ , if  $\kappa$  only slightly exceeds 2 or/and  $\beta$  is large. Note that in this case, the small argument limit of the Bessel function,  $x \ll 2$ , which yields a linear asymptotics for  $Y$ , is achieved only at a very large  $Z$ . For example, if  $\kappa = 2.5$ , the above limit is achieved only at  $Z \gg 16\beta^2$ . In order to see what happens at a smaller  $Z$ , let us assume that

$$\frac{\sqrt{\beta}}{\kappa-2} \gg 1. \quad (101)$$

Then the argument of the Bessel function in Eq. (97) is large at

$$Z \ll \left[\frac{2\sqrt{\beta}}{\kappa-2}\right]^{2/(\kappa-2)}. \quad (102)$$

In this case, one can use the large argument asymptotics of the Bessel function in Eq. (97), which leads to (cf. Eq. (82))

$$w = \sqrt{\frac{\kappa-2}{\pi}} \left(\frac{Z^\kappa}{\beta}\right)^{1/4} \cos S; \quad S = \frac{2\sqrt{\beta}}{\kappa-2} Z^{1-\kappa/2} - \frac{\kappa}{\kappa-2} \frac{\pi}{4}; \quad (103)$$

$$Y = \sqrt{\frac{\kappa-2}{\pi}} \left(\frac{Z^\kappa}{\beta}\right)^{1/4} \left[ \frac{1}{C_1} \cos^2 S + C_1 \left( C_2 \cos S + \frac{\pi}{2-\kappa} \sin S \right)^2 \right]^{1/2} \quad (104)$$

So the solution obtained for the case  $\kappa < 2$  could be extended to  $\kappa$  slightly above 2 but only in a limited range of  $Z$ . As in the case  $\kappa < 2$ , the solution describes smooth expansion

$$Y = \left(\frac{Z^\kappa}{\beta}\right)^{1/4}; \quad (105)$$

only if the constants are specially chosen,

$$C_1 = (\kappa - 2)/\pi; \quad C_2 = 0. \quad (106)$$

Generally long wavelength oscillations are superimposed on the overall expansion.

One sees that if the condition (101) is fulfilled, the flow is collimated according to Eqs. (104) or (105) in the region (102); at a larger  $Z$ , the flow becomes radial preserving the acquired collimation angle. In Fig. 1, solutions to the governing equation are shown for  $\kappa = 2.5$ ,  $\beta = 5$ . These solutions are obtained by numerical integration of Eq. (53), which is easier than numerical evaluation of integrals in Eq. (60). The smoothly expanded solution is shown by solid line whereas dashed line represents a solution with oscillations in the region (102). All the solutions go to a linear function at large  $Z$  where the flow is already well collimated. The final collimation angle could be obtained from Eq. (100) as  $\Theta = Y/Z$ . Choosing the constant  $C_1$  from Eq. (106) corresponding to the smoothly expanded solution (105), one finds the final collimation angle as

$$\Theta = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{\kappa - 1}{\kappa - 2}\right) \left(\frac{(\kappa - 2)^\kappa}{\beta}\right)^{1/[2(\kappa - 2)]}. \quad (107)$$

One sees that the collimation angle rapidly increases with increasing  $\kappa$  and decreasing  $\beta$ ;  $\Theta = 0.01/\beta^{2.5}$  at  $\kappa = 2.2$ ,  $\Theta = 0.2/\beta$  at  $\kappa = 2.5$  and  $\Theta = 0.56/\sqrt{\beta}$  at  $\kappa = 3$ . This means that if the external pressure decreases faster than  $\mathcal{P} \propto Z^{-3}$ , a narrow jet could not be produced unless the flow has already been collimated in the near zone,  $Z \sim 1$ .

Note that the solution (105) corresponds to the equilibrium collimation because it could be obtained by neglecting the first term in the governing equation. In order to figure out the collimation type of the radial flow (100), one has to find the curvature of the field surface. Expanding the solution to the governing equation (53) to higher order terms in  $1/Z$  (this could be easier done by making expansion in the equation than by expanding the general solution (60) and (97)), one gets

$$Y = c_1 Z + c_2 + \begin{cases} \frac{\beta c_1}{(3 - \kappa)(\kappa - 2)} Z^{3 - \kappa}; & 2 < \kappa < 4; \\ (2c_1^3 Z)^{-1}; & \kappa > 4; \end{cases} \quad (108)$$

where  $c_1$  and  $c_2$  are constants, which could be expressed via the constants  $C_1$  and  $C_2$  in the general solution (60). Note that  $c_1$  is in fact the final opening angle of the jet,  $c_1 = \Theta$ ; below we will use the expression (107) for  $c_1$ . One sees from Eq. (108) that at  $\kappa > 4$ , the curvature is independent of the parameters of the external pressure,  $\beta$  and  $\kappa$ , which means that the flow is not confined by the medium with such a sharply decreasing pressure. Below we do not consider this case. At  $\kappa < 4$ , the ratio of the first to the second term in the governing

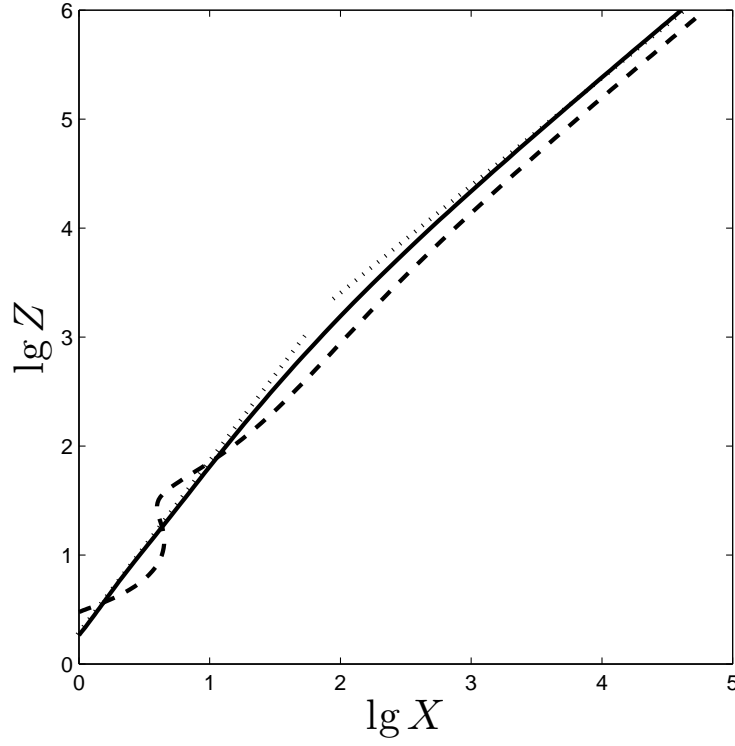


Fig. 1.— The shape of the jet,  $Y(Z)$ , for  $\kappa = 2.5$ ;  $\beta = 5$ . The solution without oscillations is shown by solid line; dashed line shows a solution with oscillations. Dotted lines show asymptotics,  $X \propto Z^{\kappa/4} = Z^{5/8}$  and  $X \propto Z$ , correspondingly.

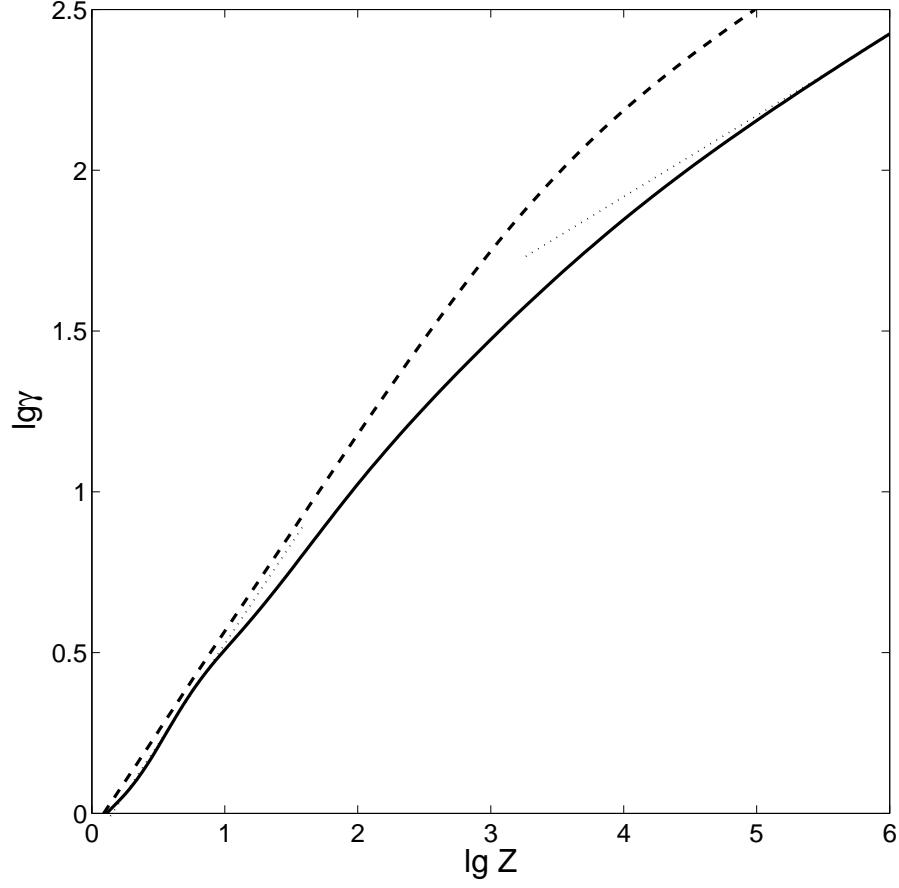


Fig. 2.— Lorentz factor of the flow shown by solid line in Fig. 1 (expansion without oscillations). Solid line shows the Lorentz factor of the flow at the boundary of the jet,  $\Psi = \Psi_0$ ; the dashed line is for the Lorentz factor at the flux surface  $\Psi = 0.2\Psi_0$ . Dotted lines show asymptotics,  $\gamma \propto Z^{\kappa/4} = Z^{5/8}$  and  $\gamma \propto Z^{(\kappa-2)/2} = Z^{1/4}$ , correspondingly.

equation (53) is

$$-Y^3 \frac{d^2 Y}{d^2 Z} = \beta \Theta^4 Z^{4-\kappa}; \quad (109)$$

which means that the flow is in the non-equilibrium regime at

$$Z > Z_1 = (\beta \Theta^4)^{-1/(4-\kappa)} \quad (110)$$

For  $\kappa = 2.5$ , the transition occurs at  $Z_1 = 70\beta^2$ .

The Lorentz factor of the flow is obtained, as before, from Eq. (66) or, for the linear  $\mu(\Psi)$ , from Eq. (75). At the condition (102), when the flow is collimated according to Eq. (105), the second term in the right-hand side dominates the first one (the equilibrium collimation) so that one comes again to Eq. (85), which is reduced, at  $\Psi \ll \Psi_0$ , just to  $\gamma = X$ . At  $Z > Z_1$ , where the flow is already nearly radial, the first term becomes dominating, which yields

$$\gamma = \frac{\eta\mu}{\Omega\Theta\alpha^{1/4}} \left( 2\beta \int_{\tilde{\Psi}}^{\Psi} \eta\mu\Phi^2 d\Psi \right)^{-1/2} Z^{(\kappa-2)/2}. \quad (111)$$

This is in accord with the general non-equilibrium scaling  $\gamma \propto \sqrt{\mathcal{R}/r}$ . For linear dependence of  $\mu$  on  $\Psi$ , this relation is reduced, with the aid of Eqs. (50) and (72), to

$$\gamma = \frac{3^{1/4}}{\Theta} \sqrt{\frac{\Psi_0}{\beta\Psi}} Z^{(\kappa-2)/2}. \quad (112)$$

In Fig. 2, we show the Lorentz factor of the smoothly expanded (without oscillations) flow at the same parameters as in Fig. 1. One sees the transition from a relatively rapid acceleration in the equilibrium regime,  $\gamma \propto Z^{\kappa/4} = Z^{5/8}$  to the slow non-equilibrium acceleration,  $\gamma \propto Z^{(\kappa-2)/2} = Z^{1/4}$ .

An important point is that in the non-equilibrium regime, the Lorentz factor increases with decreasing  $\Psi$  so that the flow is faster inside the jet than at the periphery. At small enough  $\Psi$ , the second term in Eq. (75) could become dominating, the transition occurring at

$$\frac{\Psi}{\Psi_0} = \left( \frac{Z_1}{Z} \right)^{(4-\kappa)/2}. \quad (113)$$

Recall that the non-equilibrium zone appears only at  $Z > Z_1$ . In the coordinate space, the transition occurs at

$$X = \left( \frac{Z^\kappa}{\beta} \right)^{1/4}. \quad (114)$$

Inside the zone bounded by this surface, the flow is accelerated as in the equilibrium case,  $\gamma = X$ .

One has to stress that at  $Z > Z_1$ , the Lorentz factor of the flow is determined by the curvature of the flux surfaces (non-equilibrium regime) even though the flux surfaces are nearly conical so that the Lorentz factor depends on small deviations from the conical shape. In this case, accuracy of the governing equation could become insufficient in order to find the Lorentz factor of the flow. The governing equation was obtained by neglecting the kinetic energy term in the Bernoulli equation (27) therefore the shape of the flux surfaces,  $r(\Psi, z)$ , is determined to within a factor of  $\gamma/\mu$ . In the case of the equilibrium collimation,  $\gamma \propto r$ , this accuracy is sufficient while the jet remains Poynting dominated. In the case of non-equilibrium collimation, the Lorentz factor goes as  $\sqrt{\mathcal{R}/r}$  and if the flow lines become nearly straight, the curvature could be determined by the neglected terms of the order of  $\gamma/\mu$ . In this case, the scalings (111) and (112) cease to be valid when the flow is still Poynting dominated. We address this issue in sect. 7.2.

#### 6.4. Comparison to previous works

Recently magnetic acceleration of externally confined jets was carefully studied, both numerically and analytically, by Tchekhovskoy et al. (2008) and Komissarov et al. (2008). Tchekhovskoy et al. (2008) used the force-free approximation whereas Komissarov et al. (2008) solved the full set of relativistic MHD equations. For analytical estimates, Tchekhovskoy et al. (2008) assumed that the shape of the flux surface is a power law and then found from the transfield equation the appropriate Lorentz factor and the external pressure. They obtained the scaling (83) and claimed that it is universal. Komissarov et al. (2008) based on the asymptotic transfield equation obtained by Vlahakis (2004), which is equivalent to our Eq. (34). Order of magnitude estimate of terms in this equation led them to Eq. (53) for the jet radius. Analyzing this equation, they revealed that the scaling (83) is valid only for  $\kappa < 2$ . For  $\kappa = 2$ , they obtained the scalings (88) and (89) for  $\beta > 1/4$  and  $\beta < 1/4$ , correspondingly. For  $\kappa > 2$ , they obtained the radial asymptotics.

Our approach generalizes these findings permitting the asymptotically exact solutions describing the full structure of the jet. Going beyond the simplest power law scalings also permit us to find important new qualitative features of the flow. In particular, we found that in the case of the equilibrium collimation,  $\kappa < 2$ , oscillations could be superimposed on the general expansion of the jet. For  $\kappa > 2$ , we see the transition between the equilibrium and non-equilibrium regimes, which could not be described by a power law scaling. Namely, if  $\kappa$  only slightly exceeds 2, the flow is collimated according to the equilibrium law (83) but only till some limiting distance beyond which the flow becomes radial preserving the acquired collimation angle. The larger  $(\kappa - 2)$ , the earlier (at a smaller distance) the flow becomes

radial so that at  $\kappa > 3$  the flow is practically radial from the very origin.

Tchekhovskoy et al. (2008) numerically simulated jets with different profiles of the external pressure. They found excellent agreement with the scaling (83) at  $\kappa = 2$ . For  $\kappa = 2.5$  they reported noticeable deviations from this scaling at large distances; we suppose that they observed the transition to the radial flow. At last for  $\kappa = 2.8$ , they found a wide conical jet. These results agree with our conclusions.

## 7. Terminal Lorentz factor and collimation angle.

It was shown in the previous section that if the Poynting dominated outflow is confined by the external pressure, the flow is collimated and the plasma is accelerated so that eventually the kinetic energy of the plasma could not be neglected any more. An important point is that the closer to the axis, the earlier (at a smaller  $z$ ) this happens. Note that at  $\Psi \lesssim \tilde{\Psi}$ , the Poynting flux is relatively not large from the very beginning. In this section, we address saturation of the acceleration and estimate the terminal Lorentz factor and collimation angle. For the estimates, we assume that the energy integral is described by the linear function (24); then the parameters of the flow are given by Eqs. (50), (72) and (73). It is also convenient to introduce the maximal achievable Lorentz factor,

$$\gamma_{\max} = \mu(\Psi_0) \approx \gamma_{\text{in}} \Psi_0 / \tilde{\Psi}; \quad (115)$$

which is just Michel’s magnetization parameter (Michel 1969).

The results of the previous section have been obtained in the limit  $\gamma \ll \mu$ , which means that the shape of the flux surfaces has been found with the accuracy of  $\gamma/\mu$ . We showed that the flow is accelerated as  $\gamma \sim X$  in the case of equilibrium collimation, i.e. under the condition (39), and as  $\gamma \propto \sqrt{\mathcal{R}/r}$  in the opposite case. A small error in the shape of the flux surface yields the error of the same order in the equilibrium scaling  $\gamma \propto X$  therefore this scaling could be safely extrapolated up to  $\gamma \sim \mu \sim \gamma_{\text{in}} \Psi / \Psi_0$ , i.e. until the flow ceases to be Poynting dominated. In the case of non-equilibrium collimation, one has to analyze how corrections of the order of  $\gamma/\mu$  could alter the curvature,  $1/\mathcal{R} = -d^2r/dz^2$ . Small corrections to the shape of the flux surface could significantly modify the curvature if the flux surfaces are close to cones, i.e. if  $r(Z)$  is close to a linear function. This indeed happens when the confining pressure decreases faster than  $z^{-2}$ . Therefore we consider separately the cases  $\kappa \leq 2$  and  $\kappa > 2$ .

### 7.1. The case $\kappa \leq 2$ ; transition to $\sigma \sim 1$ .

In the case  $\kappa < 2$  the Lorentz factor of the flow increases according to the equilibrium law  $\gamma = X$ . Corrections of the order of  $\gamma/\mu \ll 1$  could not affect significantly this scaling therefore the flow is accelerated up to  $\gamma \sim \mu \sim \gamma_{\max} \Psi/\Psi_0$ ; this occurs at  $X \sim \gamma_{\max} \Psi/\Psi_0$ . Making use of Eq. (74) and (83), one finds the corresponding distance as

$$Z = \left[ \frac{\beta}{3} \gamma_{\max}^4 \left( \frac{\Psi}{\Psi_0} \right)^2 \right]^{1/\kappa}. \quad (116)$$

Reverting this expression, one finds the boundary of the moderately magnetized,  $\sigma \sim 1$ , core as

$$\frac{\Psi_{\text{core}}}{\Psi_0} = \sqrt{\frac{3}{\beta}} \frac{Z^\kappa}{\gamma_{\max}^2}. \quad (117)$$

In the coordinate space, the boundary of the moderately magnetized core is found as

$$X_{\text{core}} = \sqrt{\frac{3}{\beta}} \frac{Z^{\kappa/2}}{\gamma_{\max}}. \quad (118)$$

One sees that the closer to the axis, the smaller the distance where the acceleration saturates so that a mildly magnetized,  $\sigma \sim 1$ , core expands with the distance occupying a progressively larger fraction of the jet body until the  $\sigma \sim 1$  state is achieved across the whole jet. This happens at the distance

$$Z_\sigma = \left( \frac{\beta}{3} \gamma_{\max}^4 \right)^{1/\kappa} \quad (119)$$

from the origin. At this distance, the collimation angle,  $\Theta = dY/dZ$ , is

$$\Theta = \frac{3^{(4-\kappa)/4\kappa} \kappa}{4\beta^{1/4}} \gamma_{\max}^{-(4-\kappa)/\kappa}. \quad (120)$$

When the flow ceases to be Poynting dominated, the collimation angle decreases further. It will be shown elsewhere (Lyubarsky, in preparation) that if  $\kappa < 2$ , the flux surfaces become cylindrical at the infinity and the Poynting flux is totally transferred to the kinetic energy.

If the density decreases not as a power law (but slower than  $z^{-2}$ ), one can write general estimates making use of Eq. (84) and (78). Specifically one finds that the pressure should decrease at least by a factor of  $\beta \gamma_{\max}^4$  in order for the Poynting flux to be converted into the kinetic energy of the flow. If the environment at large distances from the compact object

has the finite pressure  $p_{\text{amb}} > p_0/(\beta\gamma_{\text{max}}^4) \sim B_0^2/(6\pi\gamma_{\text{max}}^4)$ , the flow becomes cylindrical and the Lorentz factor is saturated at the value

$$\gamma_t = \left( \frac{B_0^2}{6\pi p_{\text{amb}}} \right)^{1/4}. \quad (121)$$

The above estimates are illustrated by a sketch in Fig. 3. The curve 1 shows the distribution of the Lorentz factor across the jet at a not very large distance from the origin where the flow is still Poynting dominated everywhere with except of the region  $\Psi \lesssim \tilde{\Psi}$ . Further out of the origin, the  $\sigma \sim 1$  core expands within the jet. The curve 2 shows the Lorentz factor at some intermediate distance where the core is already developed but the main body of the jet is still Poynting dominated. The curve 3 sketches the distribution of the Lorentz factor at the distance  $Z \sim Z_\sigma$  when the whole jet ceases to be Poynting dominated.

The above estimates could also be applied to the case  $\kappa = 2$ ,  $\beta > 1/4$  when the flow follows scalings obtained for  $\kappa < 2$ ; one has just substitute  $\kappa$  by 2 and  $\beta$  by  $\beta - 1/4$ . For example, the boundary of the moderately magnetized core is now presented as

$$X_{\text{core}} = \sqrt{\frac{3}{\beta - 1/4}} \frac{Z}{\gamma_{\text{max}}}; \quad (122)$$

whereas the distance where the jet ceases to be Poynting dominated is

$$Z_\sigma = \sqrt{\frac{\beta - 1/4}{3}} \gamma_{\text{max}}^2. \quad (123)$$

On the contrary, if  $\kappa = 2$  and  $\beta < 1/4$ , the flow exhibits quite different behavior because it is collimated in the non-equilibrium regime. In such a flow, the Lorentz factor grows  $\propto X$  only if and while the flow line remains close enough to the axis where the flow is in the equilibrium. After the flow line crosses the boundary between the equilibrium and non-equilibrium zone given by Eq. (96), the acceleration proceeds slower, the Lorentz factor being given by Eq. (93). In the Poynting dominated domain, the Lorentz factor is not monotonic across the jet; at a fixed  $Z$ , it increases with the radius within the equilibrium zone and decreases outwards in the non-equilibrium zone. Therefore at any  $Z$ , the Lorentz factor reaches the maximum value somewhere within the jet. Comparing the Lorentz factor with  $\mu(\Psi)$ , one finds that at a distance  $Z$  from the origin, the flow remains Poynting dominated only outside the boundary

$$\frac{\Psi_{\text{core}}}{\Psi_0} = \begin{cases} \sqrt{3}C^2\gamma_{\text{max}}^{-2}Z^{2k}; & Z < [\gamma_{\text{max}}/(3^{1/4}\beta^{1/4}C^2)]^{2/(4k-1)}; \\ 3^{1/6}(\beta C^2\gamma_{\text{max}}^2)^{-1/3}Z^{(2/3)(1-k)}; & Z > [\gamma_{\text{max}}/(3^{1/4}\beta^{1/4}C^2)]^{2/(4k-1)}. \end{cases} \quad (124)$$

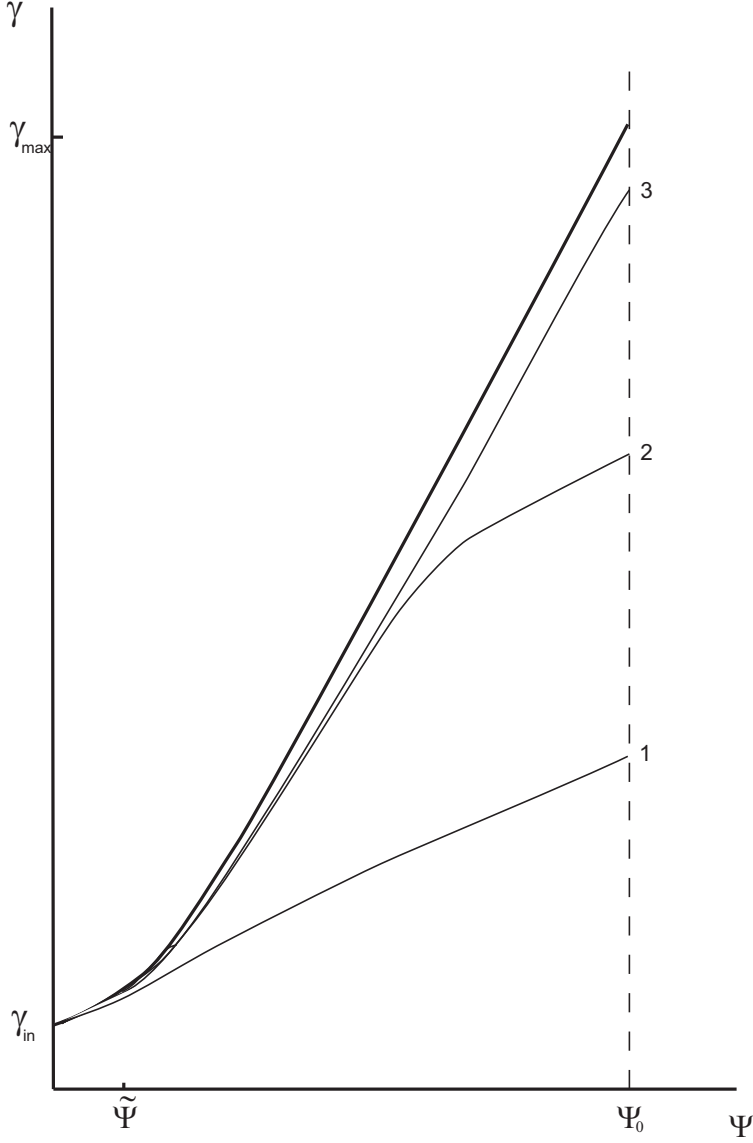


Fig. 3.— Distribution of the Lorentz factor across the jet in the case  $\kappa < 2$ ; not to scale. Thick line shows the distribution of the total energy,  $\mu(\Psi)$ . Each thin line shows the distribution of the Lorentz factor at some distance from the origin; they are labeled according to the distance, i.e. the curve 1 is the closest to the origin, the curve 3 is the farthest.

or, in the coordinates,

$$X > \begin{cases} \sqrt{3}C^2\gamma_{\max}^{-1}Z^{2k}; & Z < [\gamma_{\max}/(3^{1/4}\beta^{1/4}C^2)]^{2/(4k-1)}; \\ (3C^2\beta^{-1/2}\gamma_{\max}^{-1}Z^{2k+1})^{1/3}; & Z > [\gamma_{\max}/(3^{1/4}\beta^{1/4}C^2)]^{2/(4k-1)}. \end{cases} \quad (125)$$

One sees that the acceleration in the central part of the jet should be rapidly saturated because the Lorentz factor achieves the maximal value  $\gamma \sim \mu(\Psi)$ . Slow acceleration in the outer, non-equilibrium part of the jet continues until the Lorentz factor approaches  $\mu(\Psi)$ . The closer the flow line to the axis, the earlier (at a smaller  $Z$ ) this happens so that in this case also a moderately magnetized core expands within the body of the jet. At  $Z > [\gamma_{\max}/(3^{1/4}\beta^{1/4}C^2)]^{2/(4k-1)}$ , a maximal Lorentz factor is achieved at the boundary of the core because in the non-equilibrium jet, the acceleration is faster at the flow lines closer to the axis. The whole jet ceases to be Poynting dominated at the distance

$$Z_{\text{conv}} = \left(3^{-1/4}\sqrt{\beta}C\gamma_{\max}\right)^{1/(1-k)}. \quad (126)$$

The corresponding collimation angle is

$$\Theta = \frac{3^{1/4}k}{\sqrt{\beta}\gamma_{\max}}. \quad (127)$$

The distribution of the Lorentz factor across the jet in the case  $\kappa = 2$ ;  $\beta < 1/4$  is sketched in Fig. 4. The curve 1 shows the Lorentz factor not far from the origin. An internal part of the jet is collimated in the equilibrium regime and the Lorentz factor increases outwards from the axis. In the main body of the jet, the collimation is non-equilibrium and the Lorentz factor decreases outwards so that the Lorentz factor is a maximum at the boundary between the equilibrium and non-equilibrium zones. As the jet propagates, the flow in the internal parts reaches the  $\sigma \sim 1$  state and stops accelerating. The curve 2 shows the distribution of the Lorentz factor at the distance  $Z > [\gamma_{\max}/(3^{1/4}\beta^{1/4}C^2)]^{2/(4k-1)}$  where the flow outside the core is non-equilibrium so that the maximal Lorentz factor is achieved at the boundary of the core. The curve 3 shows the distribution of the Lorentz factor at  $Z \sim Z_{\text{conv}}$  when the whole jet ceases to be Poynting dominated.

## 7.2. The case $\kappa > 2$ ; transition to logarithmic acceleration.

At  $\kappa > 2$ , the flow is collimated only if the condition (101) is fulfilled. In this case, the flow becomes conical still being Poynting dominated; the final collimation angle is given by Eq. (107). In the conical part of the jet, the Lorentz factor grows according to Eq. (112),

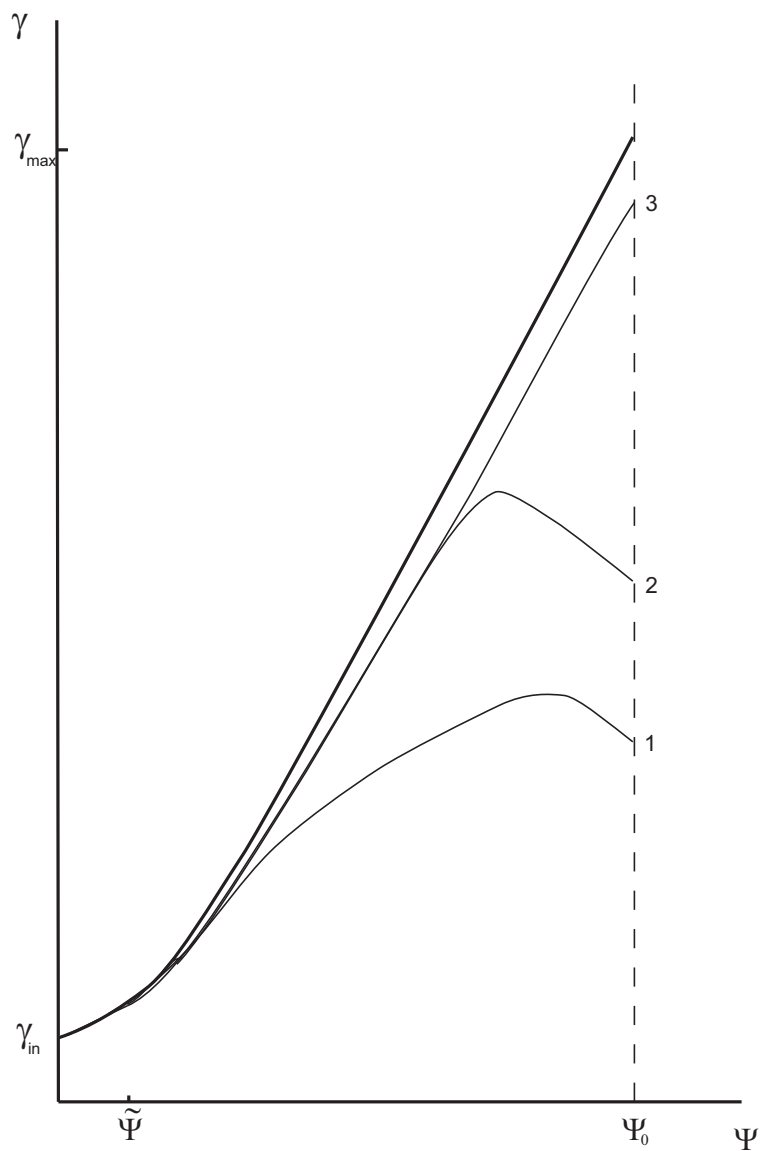


Fig. 4.— The same as in Fig. 3 but in the case  $\kappa = 2$ ,  $\beta < 1/4$ .

which corresponds to the general non-equilibrium scaling  $\gamma \propto \sqrt{\mathcal{R}/r}$ . An important point is that the curvature of the flow lines is determined by small deviations from the straight line, see Eq. (108), therefore Eq.(112) is valid only if the shape of the flow lines could be found from the governing equation with the necessary accuracy. Let us consider more carefully the jet with nearly straight flux surfaces.

In the previous sections, we found the jet structure neglecting  $\gamma$  as compared with  $\mu$  in the Bernoulli equation (27). Then the shape of the flux surfaces is presented as (see Eqs. (61) and (74))  $X(\Psi, Z) = \alpha^{1/4}\Phi(\Psi)Y(Z) = 3^{1/4}\sqrt{\Psi/\Psi_0}Y(Z)$ , where  $Y(Z)$  satisfies the governing equation. We can find limits of applicability of Eq. (112) for the Lorentz factor of the flow by substituting this equation into the Bernoulli equation (27) and finding the corresponding corrections to the shape of the flux surfaces. Eq. (112) is valid while the curvature due to this corrections remains small as compared with the curvature obtained from the solution (108) of the governing equation.

Let us present the shape of the flux surfaces as (cf. Eq. (61))

$$X = \alpha^{1/4}\Phi Y(1 + \delta); \quad (128)$$

where  $\delta(\Psi, Z) \ll 1$  describes corrections to the shape of the flux surfaces due to a non-zero  $\gamma/\mu$ . Substituting this into (27) and linearizing with respect to small  $\delta$  and  $\gamma/\mu$ , one gets

$$\frac{\partial \delta}{\partial \Psi} = \frac{\Omega^2 \gamma}{\eta \mu^2}. \quad (129)$$

Assuming for simplicity that the energy integral is described by the linear function (24), one writes in the dimensionless form

$$\frac{\partial \delta}{\partial S} = \frac{\gamma}{2\gamma_{\max} S^2}; \quad (130)$$

where

$$S = \Psi/\Psi_0. \quad (131)$$

With  $\gamma$  from Eq. (112), one finds

$$\delta = -\frac{3^{1/4} Z^{(\kappa-2)/2}}{2\Theta\sqrt{\beta}\gamma_{\max} S^{3/2}}. \quad (132)$$

Substituting this expression into Eq. (128) and differentiating twice with respect to  $Z$ , one finds the curvature of the flux surface as

$$\frac{d^2 X}{dZ^2} = 3^{1/4} \sqrt{\frac{\Psi}{\Psi_0}} \left( \frac{d^2 Y}{dZ^2} - \frac{3^{1/4} \kappa (\kappa - 1)}{4\sqrt{\beta}\gamma_{\max} S^{3/2} Z^{2-\kappa/2}} \right). \quad (133)$$

Here we take into account that  $Y \approx \Theta Z$ . The Lorentz factor of the flow could be determined from the solution to the governing equation only if the second term in brackets is small as compared with the first one. Finding  $d^2Y/dZ^2$  from Eq. (108), one sees that this is the case only at distances smaller than

$$Z_t(\Psi) = \left[ \frac{4\gamma_{\max}\Theta}{3^{1/4}\kappa(\kappa-1)} \right]^{2/[3(\kappa-2)]} \left( \frac{\beta\Psi}{\Psi_0} \right)^{1/(\kappa-2)}. \quad (134)$$

At this distance, the flow acquires the Lorentz factor

$$\gamma_t = \left( \frac{4\sqrt{3}}{\kappa(\kappa-1)} \frac{\gamma_{\max}}{\Theta^2} \right)^{1/3}. \quad (135)$$

Note that this Lorentz factor is the same for all flux surfaces whereas  $Z_t$  increases towards the periphery of the jet. This is because in the non-equilibrium regime, the acceleration rate decreases outwards from the axis.

Let us now find the Lorentz factor of the flow at  $Z > Z_t$ . With this purpose, one has to solve the transfield and the Bernoulli equations without neglecting  $\gamma$  in the Bernoulli equation. In the case of interest, the collimation is non-equilibrium therefore one can take the transfield equation in the form (42). As the flow lines are nearly straight, we can look for the solution in the form (128) with  $Y(Z) = \Theta Z$ . We again assume for simplicity that the energy integral is a linear function (24); then  $\Phi$  and  $\alpha$  are given by Eqs. (50) and (72), correspondingly. Now the transfield equation is written in the limit  $\delta \ll 1$ ,  $\gamma/\mu \ll 1$  as

$$-\sqrt{3}\Theta^2 Z \left( 2\frac{\partial\delta}{\partial Z} + \frac{\partial^2\delta}{\partial Z^2} \right) = \frac{2}{S\gamma} \frac{\partial}{\partial S} \frac{S}{\gamma}. \quad (136)$$

Linearization of the Bernoulli equation in small  $\delta$  and  $\gamma/\mu$  yields Eq. (130).

Eliminating  $\delta$  from these two equations (by differentiating Eq. (136) with respect to  $S$  and substituting Eq. (130)), one gets a single equation for  $\gamma$ :

$$\frac{\sqrt{3}\Theta^2}{4\gamma_{\max}} \gamma^2 \left( 2Z \frac{\partial\gamma}{\partial Z} + Z^2 \frac{\partial^2\gamma}{\partial Z^2} \right) = 1 + 2\frac{S}{\gamma} \frac{\partial\gamma}{\partial S} + S^2 \gamma^2 \frac{\partial}{\partial S} \left( \frac{1}{\gamma^3} \frac{\partial\gamma}{\partial S} \right). \quad (137)$$

As an initial condition, one can take  $\gamma = \gamma_t$  at  $Z = Z_t$  (the above estimates give in fact  $\gamma \sim \gamma_t$  at  $Z \sim Z_t$ ). Note that  $\gamma_t$  is independent of  $S$ . Assuming that beyond  $Z_t$ , the solution is also independent of  $S$ , one finds with the logarithmic accuracy, i.e. in the limit  $\ln Z \gg 1$ ,

$$\gamma = \left[ \frac{2\sqrt{3}\gamma_{\max}}{\Theta^2} (\ln CZ) \right]^{1/3}. \quad (138)$$

One sees that with the constant  $C = 1/Z_t$ , this function goes to  $\gamma \sim \gamma_t$  at  $Z \sim Z_t$  and still satisfies Eq. (137) with the logarithmic accuracy (because it only logarithmically depends on  $S$ , via  $Z_t$ ). So the final solution at  $Z \gg Z_t$  (in fact at  $\ln Z/Z_t \gg 1$ ) is written as

$$\gamma = \left( \frac{2\sqrt{3}\gamma_{\max}}{\Theta^2} \ln \frac{Z}{Z_t} \right)^{1/3}. \quad (139)$$

According to this solution, the flow in fact stops accelerating beyond the distance  $Z_t$  so that one can use  $\gamma_t$  as an estimate for the terminal Lorentz factor. This conclusion matches with the well-known result (Tomimatsu 1994; Beskin et al. 1998) that the radial, non-confined wind is accelerated only till  $\gamma \sim \gamma_{\max}^{1/3}$  and then the Lorentz factor grows only as  $(\ln R)^{1/3}$ .

One has to stress that according to the boundary condition (46), the flow at the boundary is accelerated till  $\gamma_{\max}$  provided the external pressure falls to zero. Therefore close enough to the boundary of the flow, the Poynting flux is efficiently converted into the kinetic energy. Specifically for the power-law pressure profile (79), the boundary condition (46) yields in the limit  $\gamma \ll \gamma_{\max}$  (24)

$$\gamma(\Psi_0, Z) = \frac{3^{1/4}}{\beta^{1/2}\Theta} Z^{(\kappa-2)/2}, \quad (140)$$

which recovers the scaling (112). One sees that at the boundary of the flow, the scaling (112) remains valid until  $\gamma \sim \gamma_{\max}$  even though in the main body of the jet, the acceleration is saturated at  $\gamma \sim \gamma_t$ . In order to find the width of the boundary region where the acceleration proceeds beyond  $\gamma_t$ , let us substitute Eq. (140) into Eq. (137) and estimate  $\partial^2 \gamma / \partial S^2 \sim \gamma / (\Delta S)^2$  necessary to satisfy the equation. This yields

$$\Delta S = \frac{\Delta \Psi}{\Psi_0} \sim \left( \frac{Z_t(\Psi_0)}{Z} \right)^{(3/4)(\kappa-2)}. \quad (141)$$

One sees that after the flow reaches  $\gamma \sim \gamma_t$  at  $Z \sim Z_t$ , the acceleration proceeds further only in a narrow region close to the boundary.

The results of this subsection are illustrated in Fig. 5. The curve 1 shows the distribution of the Lorentz factor at some distance  $Z < Z_1$  where the jet is collimated in the equilibrium regime. The curve 2 corresponds to a distance  $Z > Z_1$  where the main body of the jet is in the non-equilibrium regime so that  $\gamma$  has a maximum inside the jet, at the boundary between the equilibrium and the non-equilibrium zones. The curve 3 shows the distribution of the Lorentz factor at  $Z = Z_t(\Psi)$  for some  $\Psi < \Psi_0$ . At this distance, the acceleration is saturated in the internal part of the jet. At last the curve 4 shows the Lorentz factor at  $Z > Z_t(\Psi_0)$  where the acceleration is saturated in the main body of the jet and proceeds further only in a narrow boundary layer.

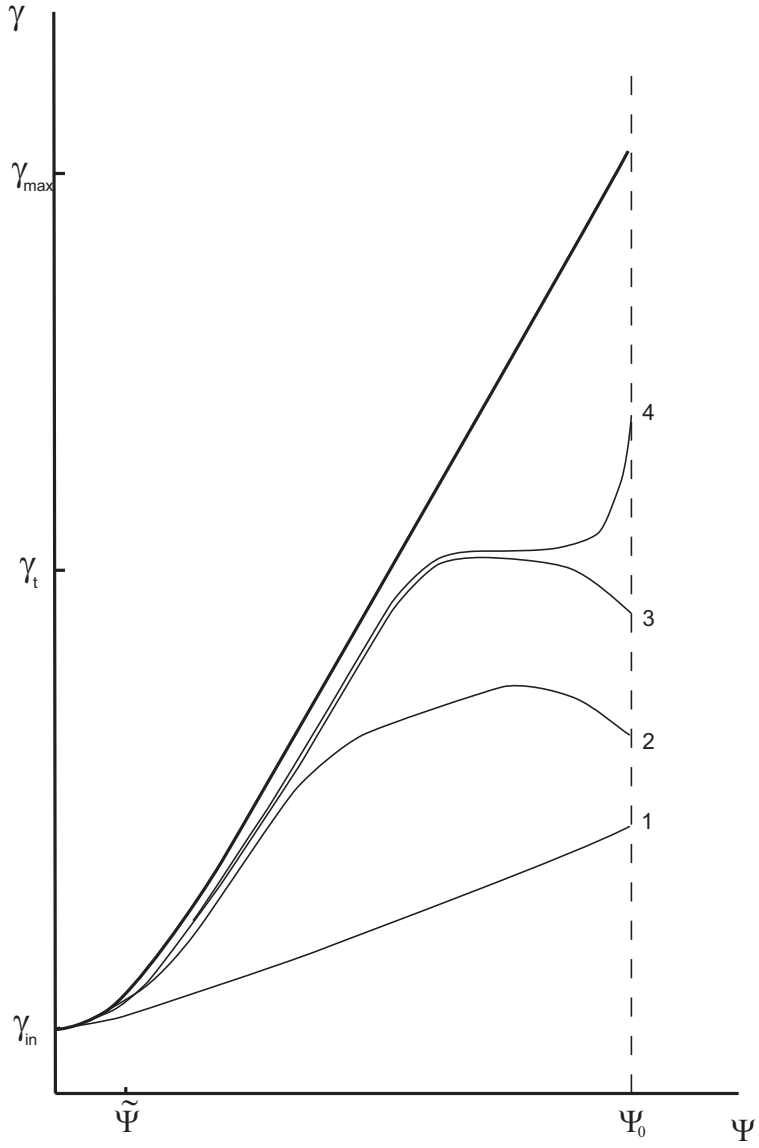


Fig. 5.— The same as in Fig. 3 but in the case  $\kappa > 2$ .

Note also that the above estimates assume that the flow remains Poynting dominated, i.e., that  $\gamma_t < \gamma_{\max} \Psi / \Psi_0$ . In the opposite case, the acceleration is saturated at  $\gamma \sim \gamma_{\max} \Psi / \Psi_0$ . In particular, if  $\gamma_t > \gamma_{\max}$ , the total Poynting flux is efficiently converted into the kinetic energy. The last condition could also be written as  $\gamma_{\max} \Theta < 1$ , which means that the flow remains causally connected in the sense that a signal sent in the transverse direction (in the proper plasma frame) could cross the jet for the proper time  $z/\gamma$ . Recall that the same condition is satisfied when  $\kappa \leq 2$  therefore in any case, the whole flow is accelerated till  $\sigma \sim 1$  only if it remains causally connected. If the acceleration is saturated at  $\gamma \sim \gamma_t < \gamma_{\max}$ , the flow is causally disconnected,  $\gamma_t \Theta \sim (\gamma_{\max} \Theta)^{1/3} > 1$ . By this reason, in particular, the flow do not "feel" the boundary any more so that the acceleration stops everywhere except of a narrow boundary region. The linkage between acceleration and causal connection of the flow was also noted by Tchekhovskoy et al. (2009). The loss of causal connection also implies the global MHD stability of such jets because global instabilities (e.g., the kink instability) has no time to develop.

Note also that  $\Theta$  is determined only by the external pressure profile whereas  $\gamma_{\max}$  is determined by the parameters of the outflow so that these two quantities are independent and any relation between them is possible. However,  $\gamma_t \Theta$  could hardly ever be very large in real systems because this quantity depends on the parameters in the power 1/3. On the other hand, such a flow could be accelerated further when and if the Poynting flux is dissipated (Thompson 1994; Lyubarsky & Kirk 2001; Drenkhahn 2002; Drenkhahn & Spruit 2002; Kirk & Skjæraasen 2003).

One should also note in this connection that according to a widely accepted view, the observed achromatic breaks in the GRB afterglow light curves occur when  $\Theta\gamma$  becomes approximately unity. Since the afterglow is attributed to the decelerating jet, this implies that  $\gamma\Theta$  was larger than unity in the prompt phase. One sees that the required property could be achieved in the MHD scenario if the confining pressure decreases with the distance something faster than  $z^{-2}$ .

## 8. The core of the jet

It was shown in the previous section that a moderately magnetized core occurs near the axis of a Poynting dominated flow so that the solutions obtained in the section 6 could not be continued to the axis. In this section, we find the structure of such a core smoothly matched with the structure of the Poynting dominated flow in the main body of the jet.

The flow near the axis is described by Eq. (40), which should be complemented by the

Bernoulli equation in the form (27):

$$\mu \left( 1 + \frac{\gamma_{\text{in}}^2}{\gamma^2} \right) - 2\gamma_{\text{in}} = (X^2 + \gamma_{\text{in}}^2) \frac{\mu - \gamma}{\Omega^2 \gamma} \frac{\partial}{\partial \Psi} \frac{\eta \mu}{\gamma}; \quad (142)$$

$$\eta(\mu - \gamma) \frac{\partial X}{\partial \Psi} = \Omega^2 X. \quad (143)$$

This set of the first order differential equations for  $X(\Psi)$  and  $\gamma(\Psi)$  should be solved at the condition  $X(0) = 0$  and matched, at large  $\Psi$ , to the solution in the main body of the flow. For example, if the main body of the flow is Poynting dominated, the solution to Eqs. (142) and (143) should be matched with the solutions obtained in Section 6. The dependence on  $Z$  enters only via this matching.

Close to the axis, the energy integral has the form of a linear function (24). Introducing the variables

$$s = 1 + \frac{\Psi}{\tilde{\Psi}}; \quad \xi = \frac{X}{\gamma_{\text{in}}}; \quad \Gamma = \frac{\gamma}{\gamma_{\text{in}}}; \quad (144)$$

one reduces Eqs. (142) and (143) to dimensionless form

$$\frac{s}{\Gamma} \frac{d\Gamma}{ds} = 1 - \frac{s + \Gamma^2(s - 2)}{2(1 + \xi^2)(s - \Gamma)}; \quad (145)$$

$$\frac{d\xi}{ds} = \frac{\xi}{2(s - \Gamma)}. \quad (146)$$

Before presenting the numerical solution to these equations, let us investigate them qualitatively.

Near the axis,  $\xi \ll 1$ , the solution is

$$s = 1 + C\xi^2; \quad \Gamma = 1 + \frac{1}{2}C\xi^4; \quad (147)$$

where  $C$  is a constant. This means, in particular, that the poloidal magnetic field is homogeneous at  $\Psi \ll \tilde{\Psi}$ .

Far from the axis,  $s \gg 1$ ,  $\xi \gg 1$ , the flow is accelerated,  $\Gamma \gg 1$ , so that Eq. (145) is reduced to

$$\frac{s}{\Gamma} \frac{d\Gamma}{ds} = 1 - \frac{s\Gamma^2}{2\xi^2(s - \Gamma)}. \quad (148)$$

The set of equations (146) and (148) is invariant with respect to the transformation  $s \rightarrow \lambda s$ ;  $\xi \rightarrow \lambda \xi$ ;  $\Gamma \rightarrow \lambda \Gamma$  so that the equations could be integrated. Namely, introducing the variables

$$u = \frac{\Gamma}{\xi}; \quad \sigma = \frac{s}{\Gamma} - 1; \quad (149)$$

(note that  $\sigma$  thus defined is indeed the ratio of the Poynting to the kinetic energy fluxes) yields the set of equations

$$2\sigma s \frac{d\sigma}{ds} = (1 + \sigma)^2 u^2; \quad (150)$$

$$2 \frac{\sigma s}{u} \frac{du}{ds} = \sigma - 1 - (1 + \sigma) u^2; \quad (151)$$

which has the first integral

$$(1 + \sigma)^2 u^2 - (\sigma - 1)^2 = c_1. \quad (152)$$

The general solution is written as

$$s = c_2 [1 - \sigma + \sqrt{-c_1}]^{1+1/\sqrt{-c_1}} [1 - \sigma - \sqrt{-c_1}]^{1-1/\sqrt{-c_1}}; \quad c_1 < 0; \quad (153)$$

$$s = c_2 [c_1 + (\sigma - 1)^2] \exp \left[ \frac{2}{\sqrt{c_1}} \arctan \frac{\sigma - 1}{\sqrt{c_1}} \right]; \quad c_1 > 0. \quad (154)$$

The solutions with  $c_1 < 0$  describes the flow with  $\sigma$  growing with the radius, and therefore with  $\Psi$ , until it reaches a constant  $\sigma_0 = 1 - \sqrt{-c_1} < 1$ . Therefore this solution represents the structure of the core in low- $\sigma$  jets. Transition to  $\sigma \rightarrow \sigma_0$  is described by the expression

$$\sigma = \sigma_0 - [2(1 - \sigma_0)]^{(2-\sigma_0)/\sigma_0} \left( \frac{c_2}{s} \right)^{-(1-\sigma_0)/\sigma_0}. \quad (155)$$

Making use of Eqs. (149) and (152), one writes this asymptotics in the original variables as

$$\frac{\gamma}{\gamma_{\text{in}}} = \frac{1}{1 + \sigma_0} \frac{\Psi}{\tilde{\Psi}}; \quad \frac{X}{\gamma_{\text{in}}} = [2(1 - \sigma_0)]^{-1/\sigma_0} c_2^{(1-\sigma_0)/2\sigma_0} \left( \frac{\Psi}{\tilde{\Psi}} \right)^{(1+\sigma_0)/2\sigma_0}. \quad (156)$$

The solutions with  $c_1 > 0$  become Poynting dominated far enough from the axis. In the limit  $\sigma \gg 1$ , one finds

$$u = 1; \quad s = c_2 \sigma^2 \exp \left( \frac{\pi}{\sqrt{c_1}} \right); \quad (157)$$

or, returning to the original variables,

$$\gamma = X; \quad \frac{\Psi}{\tilde{\Psi}} = c_2^{-1} \exp \left( -\frac{\pi}{\sqrt{c_1}} \right) \left( \frac{X}{\gamma_{\text{in}}} \right)^2. \quad (158)$$

One sees that far from the axis, the poloidal field becomes homogeneous and the solution is smoothly matched with the solution for the Poynting dominated domain, Eqs. (50), (61) and (76).

When  $c_1$  is not small, Eq. (157) implies that  $\sigma$  is large at large  $s$  so that the flow is Poynting dominated everywhere except of the region  $\Psi \lesssim \tilde{\Psi}$ . When  $c_1$  is small, the Poynting

dominated domain arises only very far from the axis, at  $s \gg \exp(\pi/\sqrt{c_1}) \gg 1$ . In the intermediate region,  $1 \ll s \ll \exp(\pi/\sqrt{c_1})$ , the solution (154) is reduced to an intermediate asymptotics

$$\sigma = 1 + \frac{c_1}{2} \ln \frac{s}{c_1}. \quad (159)$$

This means that in the core of the Poynting dominated jet,  $\sigma$  is close but remains larger than unity. It could become less than unity only when the whole jet ceases to be Poynting dominated. In the original variables, the solution in the intermediate zone is

$$\frac{\Psi}{\tilde{\Psi}} = \frac{c_1^{1/2} X}{\gamma_{\text{in}}}; \quad \gamma = \frac{1}{2} c_1^{1/2} X. \quad (160)$$

Numerical solutions to Eqs. (145) and (146) are presented in Fig. 6. For all asymptotics to be seen clearly, we plotted the curves in a very large scale. The curves are labeled by an appropriate constant  $\mathcal{C}$  in the left boundary condition (147). When  $\mathcal{C} > 1.5$ , the flow is Poynting dominated everywhere except of the region  $\Psi \lesssim \tilde{\Psi}$ . The poloidal magnetic field is homogeneous as it should be in the Poynting dominated flow with the energy integral  $\mu(\Psi)$  given by Eq. (24). In the case  $0.38 < \mathcal{C} < 1.5$ , the solution goes to the Poynting dominated asymptotics (158) only at large enough distances from the axis;  $B_p$  goes to a constant in this zone. Between  $\Psi \sim \tilde{\Psi}$  and the Poynting dominated zone, the solution is roughly described by an intermediate asymptotics (159) and (160). The poloidal magnetic field varies roughly as  $\propto 1/X$  in this zone, which means that the toroidal field,  $B_\phi = X B_p$ , and the Poynting flux remain roughly constant. At  $\mathcal{C} < 0.38$ , the solution is described, at  $\xi \gg 1$ , by the asymptotics (156), corresponding to  $\sigma = \text{const} < 1$ . In these solutions, the poloidal magnetic field decreases faster than  $1/X$ . These solutions describe the core of the jet at the stage when most of the Poynting flux is already converted into the kinetic energy.

Any solution to Eqs. (145) and (146) with the initial condition (147) describes the transverse structure of the jet at some  $Z$ . Generally  $\sigma$  decreases with  $Z$  therefore the curves in Fig. 6 describe the  $Z$  development of the jet "upside down", i.e. the upper curves describe the transverse structure of the jet at smaller  $Z$ . The moderately magnetized,  $\sigma \sim 1$ , zone occupies initially only the region  $\Psi \sim \tilde{\Psi}$ . As the distance grows, the  $\sigma \sim 1$  zone extends to a larger  $\Psi$ .

The structure of the core of the Poynting dominated jet may be found in any specific case as follows. First one finds the shape of the flux surfaces in the Poynting dominated region,  $Y(Z)$ , as it was described in the previous section. Then the transverse structure of the core at any  $Z$  is described by a solution to Eqs. (145) and (146) satisfying the left boundary condition (147) and matched, at large  $s$ , with the inner solution for the Poynting dominated flow, Eqs. (50), (61) and (72). The matching is reduced to finding an appropriate

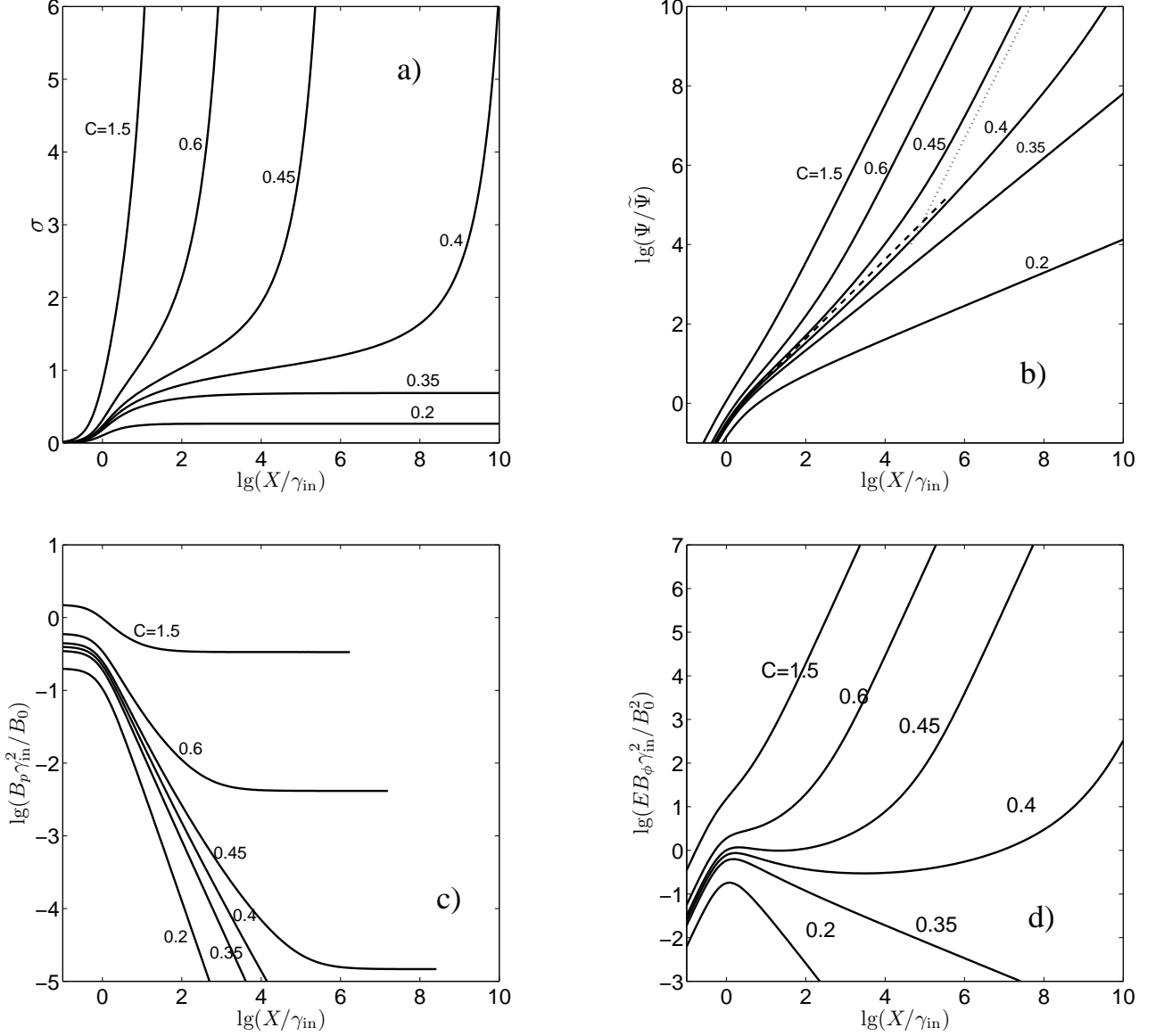


Fig. 6.— The transverse structure of the core of the jet. Shown are the ratio of the Poynting to the kinetic energy flux (a), distribution of the poloidal magnetic flux (b), the poloidal magnetic field (c) and the Poynting flux (d). Each curve describes the structure of the core at some distance  $z$ . The curves are labeled by the constant  $\mathcal{C}$  from the left boundary condition (147); the less  $\mathcal{C}$  the larger the corresponding  $z$ . The dashed and dotted lines show the asymptotics  $\Psi \propto X$  and  $\Psi \propto X^2$ , Eqs. (160) and (158), correspondingly.

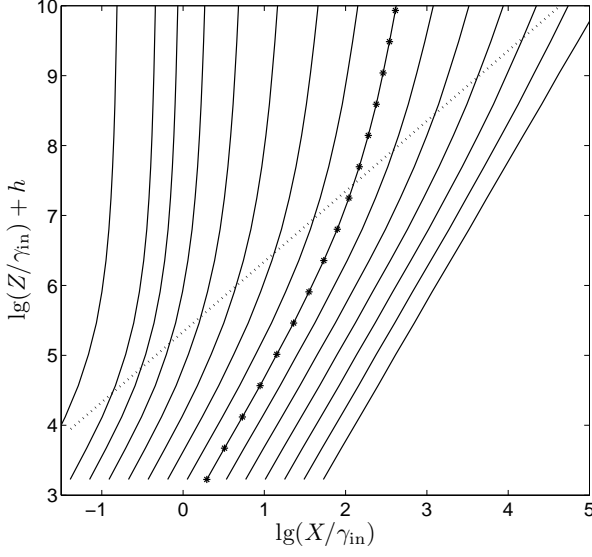


Fig. 7.— The flux surfaces at  $\kappa = 2$ ;  $\beta > 1/4$ ;  $\gamma_{\max} = 1.3 \cdot 10^5$ . The dotted line shows the boundary of the moderately magnetized core according to Eq. (122).  $h = (1/4) \lg[(\beta - 1/4)/\alpha]$

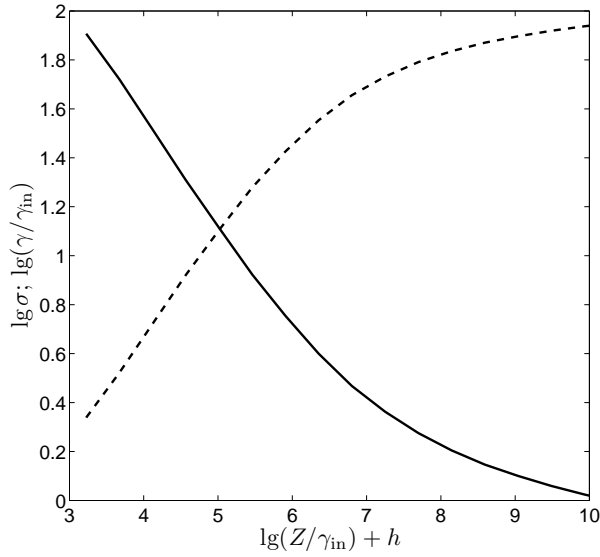


Fig. 8.— Evolution of the Lorentz factor (dashed) and of the ratio of the Poynting to the kinetic energy flux (solid) along the flux surface marked by asterisks in Fig. 7.

constant  $C$  in the left boundary condition (147), which could be done by bisection: choosing  $\mathcal{C}_1$  and  $\mathcal{C}_2$  such that the first solution goes at large  $\Psi$  to  $r$  larger than that of Eq. (61) whereas the second one goes to  $r$  smaller than that of Eq. (61), one finds the solution for  $\mathcal{C} = (\mathcal{C}_1 + \mathcal{C}_2)/2$  and continues until the necessary solution is found.

As an example, we presented in Fig. 7 the structure of the jet confined by the outer pressure  $\mathcal{P} = \beta Z^{-2}$ ;  $\beta > 1/4$ . In the Poynting dominated region, the shape of such a jet is described by Eq. (88). One sees that initially the flux surfaces within the jet diverge as  $Z^{1/2}$  together with the boundary of the jet but eventually a cylindrical core is formed (Tomimatsu 1994; Bogovalov 1995). The transition to the cylindrical flux surfaces occurs when the flow becomes moderately magnetized,  $\sigma \sim 1$ , and the Lorentz factor saturates, see Fig. 8. The transition is in fact very slow therefore the flux surfaces  $\Psi \gg \tilde{\Psi}$  become cylindrical only at extremely large distances. The further from the axis, the later the saturation is achieved. One sees from Fig.8 that while the main body of the jet remains Poynting dominated,  $\sigma$  in the core remains larger than unity just approaching unity from above, which agrees with the general analysis presented above.

## 9. Conclusions

In this paper, we developed an asymptotic theory of relativistic, magnetized jets. The study was motivated by the fact that acceleration and collimation of relativistic MHD outflows occur in a very extended zone far beyond the light cylinder. This is because the Lorentz force is nearly compensated by the electric force when the flow speed approaches to the speed of light. Because the dominant terms in the the full set of MHD equations nearly cancel each other in the far zone, it is difficult to solve them directly even numerically. In this paper, we derived asymptotic equations, which describe relativistic, steady state, axisymmetric MHD flows in the far zone. These equations could be easily solved numerically because they do not contain either intrinsic small scales like  $\Omega r$  or terms that nearly cancel each other. Moreover, in many cases one can solve them analytically or semi-analytically and find simple scalings, which provide qualitative understanding of the basic properties of relativistic MHD flows.

We applied these equations to externally confined, collimated flows. Qualitative analysis shows that there are two regimes of collimation, which we called equilibrium and non-equilibrium, correspondingly. In the first regime, the flow structure at any distance from the source is the same as the structure of an appropriate cylindrical flow. We call this regime equilibrium because in this case, the residual of the magnetic hoop stress and the electrical force is balanced, as in true cylindrical configurations, by the pressure of the poloidal magnetic field. In the non-equilibrium regime, the pressure of the poloidal field is negligibly

small so that the flow behaves as if it possesses purely azimuthal field. Such a flow could be conceived as composed from coaxial magnetic loops.

The outflow is in the equilibrium within the parabola  $r^2\Omega < z$  whereas the non-equilibrium regime occurs only outside this parabola, i.e. if the jet is not too narrow. Close enough to the axis, the flow is always in the cylindrical equilibrium. An interesting feature is that even though the pressure of the poloidal field does not hinder the collimation in the non-equilibrium regime, collimation is in fact slower in this regime than in the equilibrium one. The reason is that one can neglect the poloidal field only if the flow expands rapidly enough. In the two collimation regimes, the flow is accelerated in different ways. In the equilibrium regime, the flow Lorentz factor goes as  $\gamma \sim \Omega r$  whereas in the non-equilibrium regime, the scaling is  $\gamma \sim \sqrt{\mathcal{R}/r} \sim z/r$ .

The shape of the flux surfaces in the Poynting dominated, externally confined jet could be found by solving a simple ordinary differential equation. We studied in detail the structure of jets with a constant angular velocity confined by the external pressure with the power law profile,  $p \propto z^{-\kappa}$ . At  $\kappa \leq 2$ , the jet acquires a parabolical shape  $r \propto z^k$ , where  $k < 1$  depends on the pressure profile. The jet is collimated and accelerated until the flow ceases to be Poynting dominated. The larger the initial  $\sigma$ , the larger the final Lorentz factor of the flow. The opening angle,  $\Theta$ , decreases continuously so that the flow remains causally connected,  $\Theta\gamma \lesssim 1$ . At  $\kappa > 2$ , the flow becomes asymptotically radial. If  $\kappa$  only slightly exceeds 2, the flow still could be collimated before the flow lines become straight. The final collimation angle depends only on the pressure profile. When the flow becomes radial, the Lorentz factor could continue to grow so that the flow could become causally disconnected,  $\Theta\gamma > 1$ . However, the acceleration is practically saturated when flow reaches the terminal Lorentz factor  $\gamma_t \sim \gamma_{\max}^{1/3}\Theta^{-2/3}$ . This generalizes the well known result (Tomimatsu 1994; Beskin et al. 1998) that the non-collimated flow is accelerated practically only to  $\gamma \sim \gamma_{\max}^{1/3}$ .

The Poynting flux generally goes to zero at the axis of the flow therefore a  $\sigma \sim 1$  core is always presented in the Poynting dominated jets. We have shown that as the flow is accelerated, this core expands and the flow lines within the core approach cylinders. At  $\kappa \leq 2$ , the core expands until the  $\sigma \sim 1$  region eventually occupies the whole jet whereas at  $\kappa > 2$ , the main body of the flow remains Poynting dominated up to logarithmically large distances.

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